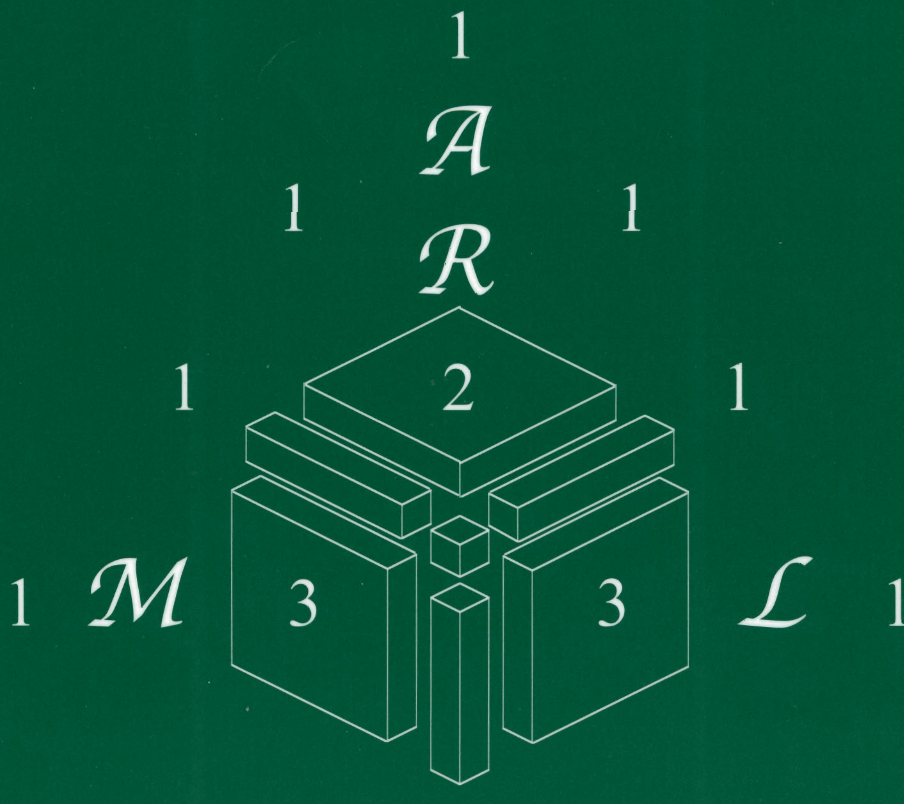
AMERICAN REGIONS MATH LEAGUE & ARML POWER CONTESTS

1995 – 2003



by Donald Barry and Thomas Kilkelly

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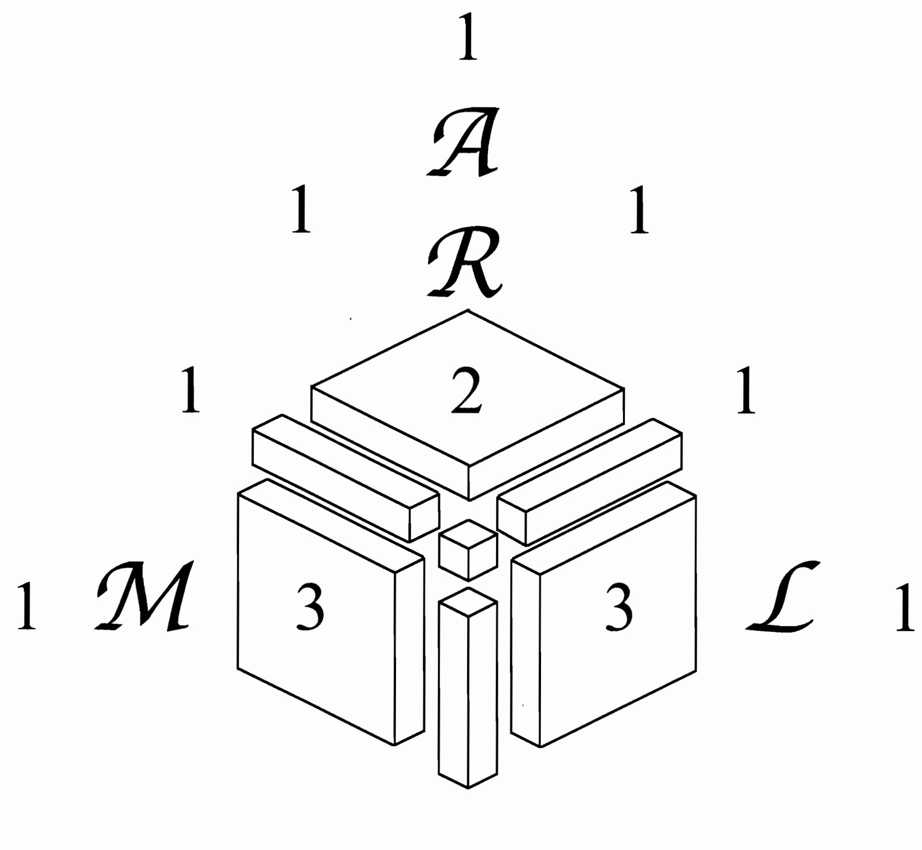

by Donald Barry and Thomas Kilkelly

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Contents

The American Regions Math League Competition

ARML is like no other mathematics contest. After months of planning and preparations, tryouts and practice sessions, busloads of students stream onto three college campuses, turning sedate institutions into beehives of excitement and anticipation. New friendships are made; old ones are renewed. Superb mathematics students from across the country are drawn together by their love of mathematics, eager to measure their abilities against other talented students as well as against a collection of truly challenging non-routine problems. ARML is distinguished by the fact that it creates communities of mathematics students and teachers, that it values and honors talented and worthy students often passed over by a society focused on media heroes. ARML's format provides for a variety of problem-solving situations and the fine problems that distinguish ARML provoke and promote mathematical insight and inventiveness, opening up new avenues of research as well as intriguing aspects of familiar material. I'll never forget the thrill I experienced the first time I saw an auditorium at Penn State packed with 900 students, all absolutely quiet, all completely intent on solving a problem. I'll never forget the explosion of joy and excitement that followed the announcement of the answer. It is an experience that repeats itself year after year, a confirmation of our hopes and dreams as teachers of mathematics.

ARML was founded in 1976 as the Atlantic Region Mathematics League. It was an outgrowth of NYSML, the New York State Mathematics League, founded in 1973, and designed to serve as a competition for the best of the teams in the math leagues in New York. Both were the joint vision of Alfred Kalfus and Steve Adrian. Alfred served as the first president of ARML and Steve, along with Joe Quartararo (from 1976 to 1982), did the organizational work, site preparations, and publicity. Marty Badoian as vice president and Eric Walstein also played crucial roles in ARML's early days. The impetus for ARML came when Marty Badoian brought a Massachusetts team to NYSML in 1975 and they did so well that an expanded competition seemed desirable. ARML was conceived as an interstate competition covering the eastern seaboard, formed through the joint action of the New York State Mathematics League, the New England Association of Mathematics Leagues, and leagues from New Jersey, Pennsylvania, Maryland, and Virginia. But ARML is a great contest and it soon attracted math students and math teachers from all over the country. By 1984 ARML was renamed the American Regions Mathematics League. Currently, it takes place simultaneously at three sites, Penn State University in State College, Pennsylvania, the University of Iowa at Iowa City, Iowa, and San Jose State University in San Jose, California. It brings together some 1500 of the finest young mathematicians in the United States and Canada. In the past teams from Russia have competed and currently teams from Taiwan and the Philippines are taking part. Taiwanese educators even created a similar competition for schools in Taiwan called TRML.

ARML is a competition between regions. A region may be as large as a state–there are teams from Texas, Minnesota, and Georgia, it may be half a state–eastern and western Massachusetts field teams, it may be a county such as Suffolk county in New York, it may be a city such as Chicago or New York, or a region may even be a school. For example, Thomas Jefferson High School in Fairfax, Virginia has sent some remarkably successful teams. Each team consists of 15 students and a region may send more than one team.

The contest consists of 6 parts. First is the Team Round. A team's 15 students are in one classroom, they receive 10 problems, they have 20 minutes to solve them as a group, and they may use calculators. Typically, problems are divided up so that every problem is being worked on by at least one student, answers are posted, and hurried consultations take place if there is disagreement. Next comes the Power Question. Teams are given 60 minutes to solve a series of in-depth questions on one topic and/or prove a number of theorems on that topic. The teams' papers are graded by a hardworking group of teachers tucked away in a corner of the auditorium. Following the Power Question, the teams come together in a large auditorium for the Individual Round. Here they solve 8 questions, given in pairs with 10 minutes for each pair, right answer only. Neither calculators nor collaboration are allowed. Initial problems are easier, but an easier problem is generally paired with a more difficult one. We try to write problems so that 80–90% of the students can solve the first one and less than 5% can solve the last one. The next round is the Relay Round. The teams of 15 are divided up into 5 groups of 3. Each 1st person in a group gets the same problem, each 2nd person gets the same problem, and each 3rd person gets the same problem. The first person's problem has all the information necessary to solve it, but the second person's problem requires the first person's answer, and the third person needs the second person's answer. None of the three knows the other's questions. A well-written relay problem enables the second and third team members to do considerable work while waiting. They may even be able to discover a candidate pool of likely answers to their problem.

Those four rounds are the only rounds that count for the Team and Individual competition. Scoring is as follows: each correct answer on the Team Round earns 4 points. Sometimes a team gets all 40 points, but 36 has typically been the top score and the average for all the teams has been between 20 and 24. The Power Question is worth 40 points; often one or more teams earn a perfect score. Each individual problem is worth 1 point per contestant, meaning that the team can earn as many as $8 \cdot 15 = 120$ points on this round. Generally, 90 is a great score. On the Relay Round, only the third person's answer is scored. If the group of 3 gets the problem correct within 3 minutes they earn 4 points, if within 6 minutes they earn 2 points. Thus, each relay is worth a maximum of $5 \cdot 4 = 20$ points; a score of 14 is quite good. The top team scores for the entire contest fluctuate between 160 and 190 points although the 1995 and 1997 contests were particularly difficult, causing top scores to drop into the 120's.

The Tiebreaker Round follows the relays. ARML only gives 3 top individual prizes and usually a playoff must take place between the top scorers. It is a dramatic moment. At each site the contenders come to the front of the auditorium, they receive the problem at the same time that it is flashed on an overhead screen. Each student is timed and the winners are determined by who gets the correct answer most quickly. Sometimes several tiebreakers must be given to determine the 3 winners. The Super Relay, a competition that first took place in 1996, is the final round and it is just for fun. It is a relay race for the whole team involving 15 questions. The team that gets the correct answer to the last question is the winner, earning a round of applause and bragging rights 'til next year. To keep the Super Relay from getting bogged down, we usually slip one or two problems into the relay that can be solved without requiring the previous person's answer. In the 1996 – 1999 Super Relays, students passed answers from position 1 to 15. Starting in 2000, students passed from both ends into position 8. That student's question required two answers in order to be solved.

ARML is quite an undertaking. It brings together great students, it is one of the few mathematics contests that involves exciting travel, meeting lots of other students, and renewing friendships established during summer programs. It relies on the work of a large number of dedicated adults who do all the organizing and spend hours finding or writing problems to use in practice sessions. It is a contest that promotes and demands creativity and imagination. The students who take part in ARML are very experienced problem solvers, quick and insightful. Those of us who write problems for ARML respect the abilities of our participants and, consequently, we spend hours developing problems that spring from high school mathematics, yet are out of the ordinary, problems that can't be attacked in rote fashion. We don't just take a theorem and write a problem that employs it. We ask questions, we imagine situations we'd never thought of before, we try out this idea and that possibility, we run into dead-ends, and we occasionally stumble across a really neat idea and that's the one that makes it into an ARML competition. There are many different topics on each year's ARML, but more important, there are a wide variety of ways of thinking.

This publication makes available the problem sets and solutions for the 1995 to 2003 ARML contests. Three earlier publications contain ARML problems from its beginning in 1976 to 1994 and NYSML problems from its beginning in 1973 to 1992. The first was *NYSML–ARML Contests 1973–1982*, published in 1983 by Mu Alpha Theta. The second was *NYSML–ARML Contests 1983–1988* by Gilbert Kessler and Lawrence Zimmerman, published by the National Council of Teachers of Mathematics in 1989. The third was *ARML –NYSML Contests 1989–1994* by Lawrence Zimmerman and Gilbert Kessler published by MathPro Press in 1995.

ARML has an executive board that organizes each contest. Alfred Kalfus was president from 1976 until 1989 when Mark Saul of Bronxville High School, NY took over. Mark retired in 2001 and Tim Sanders, director of the Great Plains Mathematics League, took over. Starting in June of 2004, J. Bryan Sullivan will be the president. Current board members include Marty Badoian, Steve Adrian, Linda Berman, J. Bryan Sullivan, John Benson, Josh Zucker, Paul Dreyer, Kristie Sallee, and Amy Gibbs. Marty, Steve, and Bryan have been involved with ARML since its inception.

Don Barry has been the head author since 1995 and during that time he has received a number of great problems, solutions, and a lot of spirited help from Paul Dreyer, Ed Early, Zuming Feng, Zac Franco, Chris Jeuell, Paul Karafiol, Rick Parris, Ashley Reiter, Leo Schneider, Ravi Vakil, Eric Wepsic, Elizabeth Wilmer, and Andrei Zelevinsky. The following teachers at Phillips Academy in Andover, Massachusetts have helped enormously in editing, reviewing, and occasionally suggesting problems: Frank Hannah, Chris Odden, David Penner, Bob Perrin, and Bill Scott. Our work has benefited from and, we hope, built upon the fine work done by ARML's previous problem writers including Steve Conrad, Dr. Norman Schaumberger, Dr. Erwin Just, Irwin Kaufman, Howard Shapiro, Harry Ruderman, Gilbert Kessler, and Larry Zimmerman.

I would like to dedicate this book to my wife, Roxy, whose cheerful patience, support and encouragement have been so invaluable and to Bob Perrin whose insights and brilliant explanations have helped make each and every contest so much better.

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The ARML Power Contest

In 1995, under the leadership of its president, Mark Saul, ARML introduced the ARML Power Contest. Modeled on the Power Question which involves, as Mark noted, "cooperative effort in exploring a problem situation through the solution of chains of related problems", ARML began to offer a mail-in competition. In November and in February, each participating team receives a set of problems based on a major theme. Some of the problems require just a numerical answer, some require justification, while others require that a proof be written. The mathematics of the problems has been geared so that students in an honors class, a math club, or on a math team can have a unique problem solving and mathematics writing experience. There is no limit to the size of the team, but the time for solving the problem set is limited to 45 minutes. Calculators are permitted, but no other outside aids are allowed. Initially, mathematicians from Thomas Watson Laboratories of IBM, the National Security Agency, and Los Alamos Laboratories graded the papers. Currently, the papers are now graded by the author and some of his colleagues. There are 40 points per contest; the team score is the sum of the scores of the two contests. A team may represent a region, not just a school, but the contest must still be taken by all members of the team at the same time and place. The contest has grown steadily. As of the 2003-2004 academic year there were 43 teams from the United States, Canada, and Bulgaria participating.

The Power Question and the Power Contest are both designed to simulate actual mathematical research activity. This is not easy. As Mark noted,

> "Power Contest problems are difficult to write. They must provide meaningful problem situations both for the novice and veteran mathletes. They must attract schools with strong traditions in mathematics competitions, yet offer experiences for students new to such events. Thus, they must build mathematically significant results out of mathematically trivial materials."

Roger Sadlowsky, a ARML coach from Minnesota, coordinated the first few years of the competition. The initial authors were ARML coach Thomas Kilkelly and ARML alumni John Tillinghast and David Miller. Two weeks in the Soviet Union, working with Soviet mathematicians, teachers, and students, provided the necessary ideas for the writers to produce the first few years of problems. This effort was underwritten by Best Practices in Education, a non-profit foundation dedicated to finding exemplary education practices around the world and adapting them for American educators and students. In 1998 Sadlowsky retired, Tillinghast and Miller went on to graduate studies and Kilkelly volunteered to take over as coordinator and author.

I would like to thank Mark Saul for his encouragement and trust, Peter Moxhay and Gail Richardson from Best Practices in Education for their vision in providing funds for the trip to Russia, John Tillinghast and David Miller for their insight and enthusiasm in getting the project started, Aloysha Belov, Grisha Komdakov, and Dmitry Fomin for their Russian hospitality and mathematical creativity, Evgenia (Jenny) Sendova (Bulgaria) and Bogdan Enescu (Romania) for their continuing enthusiasm and support, and Dr. David Cervone and Dr. Steve Heillig for their proof-reading and exchange of mathematical ideas. Special thanks to Carla, my wife, whose patience, support and encouragement were and are essential for my participation in this project. I want to say that I really enjoy the research, problem solving, mathematical writing challenges the competition provides.

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Problem Writers and Reviewers for ARML 1995 – 2003

1995: Don Barry, George Markowsky, David Penner, Bill Scott, Eric Wepsic, and Andrei Zelevinsky.
Reviewer: David Penner.

1996: Don Barry, Zac Franco, Bill Scott, Eric Wepsic, and Andrei Zelevinsky.
Reviewers: Frank Hannah, David Penner, Bob Perrin, Leo Schneider, and Bill Scott.

1997: Don Barry, Zuming Feng, Zac Franco, Ravi Vakil, Eric Wepsic, and Elizabeth Wilmer.
Reviewers: David Penner, Bob Perrin, and Andrei Zelevinsky.

1998: Don Barry, Zuming Feng, Zac Franco, Ravi Vakil, Eric Wepsic, Elizabeth Wilmer, and Andrei Zelevinsky.
Reviewers: David Penner, Bob Perrin, and Andrei Zelevinsky.

1999: Don Barry, Paul Dreyer, Zuming Feng, Zac Franco, Chris Jeuell, Ashley Reiter, Leo Schneider, Ravi Vakil, Eric Wepsic, and Andrei Zelevinsky.
Reviewers: David Penner and Bob Perrin

2000: Don Barry, Paul Dreyer, Ed Early, Noam Elkies, Zuming Feng, Zac Franco, Chris Jeuell, Ashley Reiter, Leo Schneider, Ravi Vakil, and Eric Wepsic.
Reviewers: David Penner, Bob Perrin, and Andrei Zelevinsky.

2001: Don Barry, Paul Dreyer, Ed Early, Zuming Feng, Zac Franco, Chris Jeuell, Leo Schneider, Ravi Vakil, and Eric Wepsic.
Reviewers: David Penner and Bob Perrin.

2002: Don Barry, Paul Dreyer, Ed Early, Zuming Feng, Zac Franco, Chris Jeuell, Paul Karafiol, Leo Schneider, and Eric Wepsic.
Reviewers: David Penner and Bob Perrin.

2003: Don Barry, Paul Dreyer, Ed Early, Zuming Feng, Zac Franco, Chris Jeuell, Paul Karafiol, Chris Odden, Rick Parris, Leo Schneider, and Eric Wepsic.
Reviewers: Bob Perrin and Sam Vandervelde.

Problem Writers for the ARML Power Contest

Thomas Kilkelly has written all the contests save for *Color Transformations* and *Rotating Decimal*s which were written by David Miller.

1995 Team Competition

Division A		Division B	
1. New York City A	126	1. Chicago B	82
2. Texas Gold	123	2. All Pennsylvania	75
3. Thomas Jefferson HS A	119	3. Missouri	71
		3. Rhode Island	71

1995 Individual Competition

1. Daniel Stronger	New York City A	7
2. Chris Chang	Northern California A	6
3. Michael Korn	Minnesota Gold	6

1996 Team Competition

Division A		Division B	
1. San Francisco Bay Area A	179	1. Upstate New York A	129
2. New York City A	167	2. New York City B	127
3. Thomas Jefferson HS A	158	3. Massachusetts E	125
3. Montgomery A	158		

1996 Individual Competition

1. Nathan Curtis	Thomas Jefferson A	8
2. Lorentz Shyu	San Francisco Bay Area B	8
3. Matt Gealy	Howard County	7

1997 Team Competition

Division A		Division B	
1. Minnesota Gold	125	1. Suffolk A	90
2. New York City A	116	2. Howard	89
3. Texas Gold	114	3. Georgia B	88

1997 Individual Competition

1. Davesh Maulik	Nassau A	7
2. Li Chung Chen	San Francisco Bay Area A	6
3. Mile Colsher	Wisconsin Red	6

1998 Team Competition

Division A

1. Massachusetts A 171
2. San Francisco Bay Area A 167
3. Montgomery County A 157

Division B

1. San Francisco Bay Area B 138
2. Phillips Exeter Academy 122
3. Taiwan Yale 121
4. New York City Y 116

1998 Individual Competition

1. Gabriel Carroll .... San Francisco Bay Area A 8
2. Melanie Wood .... Indiana Gold 8
3. Brian Ginsberg .... Chicago A 8

1999 Team Competition

Division A

1. San Francisco Bay Area A 187
2. Massachusetts A 181
3. Thomas Jefferson HS 178

Division B

1. Taiwan C 158
2. Mercer County 145
3. Washington Gold 140
4. Massachusetts E 137

1999 Individual Competition

1. Gabriel Carroll .... San Francisco Bay Area A 8
2. Reid Barton .... Massachusetts A 8
3. Lawrence Detler .... New York City A 8

2000 Team Competition

Division A

1. Chicago A 172
1. San Francisco Bay 172
3. New York City A 171

Division B

1. Connecticut A 127
1. Peninsula South Bay 127
3. Iowa A 125
3. Northern California 125

2000 Individual Competition

1. Tiankai Liu .... San Francisco Bay A 8
2. Gabriel Carroll .... San Francisco Bay A 8
3. Sasha Schwartz .... All Pennsylvania 8

2001 Team Competition

Division A		Division B	
1. San Francisco Bay A	191	1. Michigan Reals	160
2. Massachusetts A	185	2. Ontario B1	134
3. Thomas Jefferson HS A	175	3. Chicago C	120

2001 Individual Competition

1. Gabriel Carroll	San Francisco Bay Area A	8
2. Gregory Price	Thomas Jefferson HS A	8

2002 Team Competition

Division A		Division B	
1. Thomas Jefferson HS A	190	1. New York City S	122
2. Massachusetts A	172	2. Southern California	114
3. Chicago A	161	3. Thomas Jefferson HS B	107

2002 Individual Competition

1. Ruozhou (Joe) Jia	Chicago A	8
2. Anatoly Preygel	Montgomery County A	8
3. Jeffrey Amos	Kansas Regional	8

2003 Team Competition

Division A		Division B	
1. Thomas Jefferson HS A	155	1. Connecticut A	112
2. San Francisco Bay A	153	2. Lehigh Valley	98
3. Chicago A	147	3. Missouri A	95
		3. Nebraska	95

2003 Individual Competition

1. Anders Kaseorg	North Carolina	8
2. Jonathan Lowd	San Francisco Bay Area A	7
3. Bob Hough	Michigan Reals	7

ARML Power Contest Prize Winners

1994-95
1. Vestavia Hills H.S., Vestavia Hills, AL
2. St. Paul Academy, St. Paul, MN
3. Louisiana ARML, LA

1995-96
1. Stuyvesant H.S., New York City, NY
2. Mounds View H.S., Arden Hills, MN
3. North Carolina School of Science and Math, Durham, NC

1996-97
1. Stuyvesant H.S., New York City, NY
2. Vestavia Hills H.S., Vestavia Hills, AL
3. St. Paul Central H.S., St. Paul, MN

1997-98
1. Stuyvesant H.S., New York City, NY
2. St. Paul Central H.S., St. Paul, MN
3. Evanston H.S., Evanston, IL

1998-99
1. Lynbrook H.S., San Jose, CA
2. North Carolina School of Science and Math, Durham, NC
3. Evanston H.S., Evanston, IL

1999-00
1. Stuyvesant H.S., New York City, NY
2. Gunn Math Circle, Gunn H.S., Palo Alto, CA
2. Wayzata H.S., Wayzata, MN

2000-01
1. Phillips Exeter Academy, Exeter, NH
2. Sofia Math Circle (Bulgaria)
3. Stuyvesant H.S., New York City, NY

2001-02
1. Thomas Jefferson H.S., Fairfax, VA
2. Sofia Math Circle (Bulgaria)
3. Phillips Exeter Academy, Exeter, NH

2002-03
1. Stuyvesant H.S., New York City, NY
2. Sofia Math Circle (Bulgaria)
3. Phillips Exeter Academy, Exeter, NH

2003-04
1. St. Louis Math Circle, St. Louis, MO
2. Academy for the Advancement of Science & Technology, Hackensack, NJ
3. Honorable Vincent Massey SS, Ontario, Canada

ARML

1995

THE 20TH ANNUAL MEET

ARML celebrated its twentieth year of competition by expanding to a new, western site. The University of Nevada at Las Vegas (UNLV) joined the existing ARML sites at the University of Iowa and Penn State University on June 2nd and 3rd for the 1995 meet. New teams from Idaho, Nevada, Utah, and western Pennsylvania joined the ranks, and the former state team from California was replaced by six teams, three from northern California and three from southern California. Including five alternate teams, a total of eighty-five teams and over 1275 students participated. With the addition of the west coast site, some logistical changes were required. Specifically, the teams at UNLV did both the Power and Team rounds on Friday night, while the Iowa and Penn State teams ran the meet as always.

Susan Heicklen of Central Pennsylvania received the Samuel Greitzer Distinguished Coach Award. She has been a key figure in establishing and maintaining Penn State as a successful ARML site.

Eric Walstein, the ARML regional representative from the Mid-Atlantic states and coach of the Montgomery County, MD teams received the Alfred Kalfus Founder's Award. He has been the coach of Montgomery County for all 20 years of ARML's existence and was a primary force behind ARML's ongoing exchange with Russia.

The following received the Zachary Sobol award for outstanding contributions to their ARML teams:

Matt Ahart	North Hollywood, CA—organizer of a new team
Franz Ed Boas	San Diego, CA—organizer of a new team
Ben Stevens	San Diego, CA—organizer of a new team
Lauren Williams	South Bay, CA—organizer of a new team
Mathew Crawford	Alabama—two year captain
Scott Kempen	Wisconsin—captain and 4 year participant
Robbie McNerney	Western Massachusetts—captain and six year participant

T–1. Let $\underline{A}\,\underline{B}\,\underline{C}$ represent a three-digit base 10 number whose digits are A, B, and C with $A \geq 1$. Compute the minimum value of $\underline{A}\,\underline{B}\,\underline{C} - \left(A^2 + B^2 + C^2\right)$.

T–2. In ΔABC, side $\overline{BC}$ is the average of the other two sides. If $\cos \angle C = \dfrac{AB}{AC}$, compute the numerical value of $\cos \angle C$.

T–3. Compute the number of distinct ways in which 77 one-dollar bills can be distributed to 7 people so that no person receives less than \$10.

T–4. Starting at the same time at corner M, Alison and Ben run in opposite directions around square track $MNPQ$, each travelling at a constant rate. The first time they pass each other at corner P is the tenth time that they meet. Compute the smallest possible ratio of the faster person's speed to the slower person's.

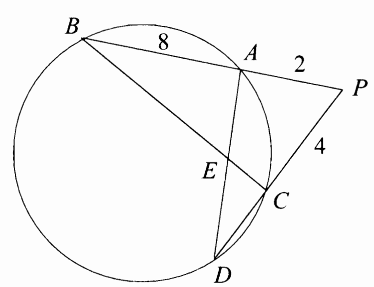
T–5. Determine all integer values of θ with $0° \leq \theta \leq 90°$ for which $(\cos\theta + i\sin\theta)^{75}$ is a real number.

T–6. Assuming the expression converges, determine the largest integer n with $n \leq 4{,}000{,}000$ for which

$\sqrt{n + \sqrt{n + \sqrt{n + \ldots}}}$ is rational.

T–7. A trapezoid has a height of 10, its legs are integers, and the sum of the sines of the acute base angles is $\dfrac{1}{2}$. Compute the largest sum of the lengths of the two legs.

T–8. Points A, B, C, and D lie on the given circle. If $AB = 8$, $AP = 2$, and $PC = 4$, determine the ratio of the area of quadrilateral $PAEC$ to the area of ΔBAE.



T–9. Points C and D lie on opposite sides of line $\overline{AB}$. Let M and N be the centroids of ΔABC and ΔABD respectively. If $AB = 25$, $BC = 24$, $AC = 7$, $AD = 20$, and $BD = 15$, compute MN.

T–10. If $\log_{10}14 = x$, $\log_{10}15 = y$ and $\log_{10}16 = z$, then determine the number of elements in

$S = \left\{\log_{10}1, \log_{10}2, \log_{10}3, \ldots, \log_{10}100\right\}$ which can be written in the form $ax + by + cz + d$ for rational numbers a, b, c, and d.

ANSWERS ARML TEAM ROUND – 1995

1. 27
2. $\frac{5}{7}$
3. 1716
4. $\frac{11}{9}$
5. 0°, 12°, 24°, 36°, 48°, 60°, 72°, 84°
6. 3,998,000
7. 441
8. $\frac{5}{16}$
9. $\frac{39}{5}$
10. 46

T–1. $100A + 10B + C - A^2 - B^2 - C^2 = \left(100A - A^2\right) + \left(10B - B^2\right) + \left(C - C^2\right)$. We minimize the sum by minimizing each term. Since $10B - B^2 = (10 - B)B \geq 0$ for $B \in \{0, 1, \ldots, 9\}$ with equality iff $B = 0$, choose $B = 0$. Since the graph of $C(1 - C)$ is a concave down parabola with vertex at $C = \frac{1}{2}$, the graph is decreasing for $C > \frac{1}{2}$ and the minimum occurs at $C = 9$. Since $100A - A^2 = A(100 - A)$ is a concave down parabola with vertex at $A = 50$, we minimize the expression by maximizing the difference between A and 50, i.e., set $A = 1$. Thus, the minimum value of the difference is $109 - \left(1^2 + 0^2 + 9^2\right) = 109 - 82 = \boxed{27}$.

T–2. Using the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos C = a^2 + b^2 - 2ab\left(\frac{c}{b}\right) = a^2 + b^2 - 2ac$. Since $a = \frac{b + c}{2}$, then $c^2 = \left(\frac{b + c}{2}\right)^2 + b^2 - \frac{2c(b + c)}{2}$. Simplifying, we obtain $7c^2 + 2bc - 5b^2 = 0 \rightarrow (7c - 5b)(c + b) = 0$, so $\frac{c}{b} = \boxed{\frac{5}{7}}$.

T–3. <u>1st Method</u>: Assume that every person has received \$10 and it remains to distribute \$7 among 7 people. Imagine the seven dollars laid out on a table. The first person puts a marker after collecting her money, the others do the same except for the last person. A distribution, for example, could be represented by the diagram and table below:

[\$] [\$] | | [\$] | | | [\$] [\$] [\$] | [\$]

person	1	2	3	4	5	6	7
receives	\$2	\$0	\$1	\$0	\$0	\$3	\$1

Hence, the number of distributions is the same as the number of placements of 7 bills and 6 bars in a line of 13. Choose 6 locations for the bars: ${}_{13}C_6 = \boxed{1716}$.

<u>2nd Method</u>: Let x_i denote the number received by the ith person for $i = 1, \ldots, 7$. The number of distributions is the number of integer solutions of the equation $x_1 + x_2 + \ldots + x_7 = 77$ for $x_i \geq 10$. Letting $x_i - 9 = y_i$, we obtain the equation $y_1 + y_2 + \ldots + y_7 = 77 - 7 \cdot 9 = 14$ with each $y_i \geq 1$.

Setting $z_1 = y_1$, $z_2 = y_1 + y_2, \ldots, z_6 = y_1 + \ldots + y_6$, we then obtain $1 < z_1 < z_2 < \ldots < z_6 \le 13$. The number of distributions is simply the number of six-tuples $\{z_1, z_2, \ldots, z_6\}$ where each six-tuple is a combination of 6 integers from 1 to 13 inclusive. Hence, ${}_{13}C_6 = 1716$.

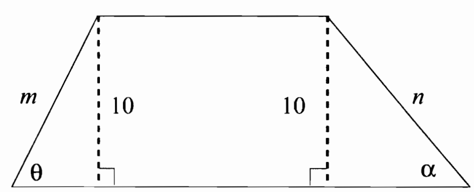
T–4. 1st Method: Without loss of generality, let the distance around the square be 400. At the first meeting the faster person will have traveled $200 + d$, the slower $200 - d$; at the 10th encounter the distances will be $2000 + 10d$ and $2000 - 10d$. The distances are equivalent to 5 circuits plus or minus 1/2 a lap or 3/2 a lap, etc. The speeds are most nearly equal if the faster person has traveled 5.5 laps and the slower person 4.5 laps. The smallest possible ratio of their speeds is $\dfrac{5.5}{4.5} = \boxed{\dfrac{11}{9}}$.

2nd Method: Let $MN + NP = 1$, let x be the distance traveled by the slower person between consecutive meetings, let $2 - x$ be the distance traveled by the faster person. Then $0 < x < 1$ and x must satisfy $10x = 2k + 1$ for some integer k. The largest possible x satisfying this equation is $x = \dfrac{9}{10}$ and the ratio is $\dfrac{2-x}{x} = \dfrac{11/10}{9/10} = \dfrac{11}{9}$.

T–5. $(\cos\theta + i\sin\theta)^{75} = \cos(75\theta) + i\sin(75\theta)$. This is real iff $\sin(75\theta) = 0$, meaning that $75\theta = 180k$ for integral k. Thus, $\theta = \dfrac{12k}{5}$ and this implies that k is divisible by 5. Let $k = 5m$, then $\theta = 12m$ and so we have the following values for θ: $\boxed{0°, 12°, 24°, 36°, 48°, 60°, 72°, \text{ and } 84°}$.

T–6. Let $x = \sqrt{n + \sqrt{n + \sqrt{n + \ldots}}}$, then $x^2 = n + x$, so $n = x^2 - x$ for x rational. But since n is an integer and $n = x(x-1)$, then x is an integer as well and if n is to be as large as possible but less than or equal to 4,000,000, then $x = 2000$, making $n = 2000 \cdot 1999 = \boxed{3{,}998{,}000}$.

T–7. Since $\sin\theta + \sin\alpha = \dfrac{10}{m} + \dfrac{10}{n}$, then $\dfrac{1}{2} = 10\left(\dfrac{n+m}{n \cdot m}\right)$ giving



$mn - 20n - 20m = 0 \rightarrow mn - 20n - 20m + 400 = 400$.

This factors as $(m-20)(n-20) = 400$. The sum $(m+n)$ is maximized when one of the factors equals 400 and the other equals 1. Set $n - 20 = 400$, giving $n = 420$ and $m - 20 = 1$, giving $m = 21$. Thus, $m + n = \boxed{441}$.

T–8. In this problem we will use the expression $a(\Delta BAE)$ to represent the area of ΔBAE.

$PA \cdot PB = PC \cdot PD \rightarrow 20 = 4 \cdot PD \rightarrow PD = 5$ and $DC = 1$. Since $\Delta BAE \sim \Delta DCE$, then

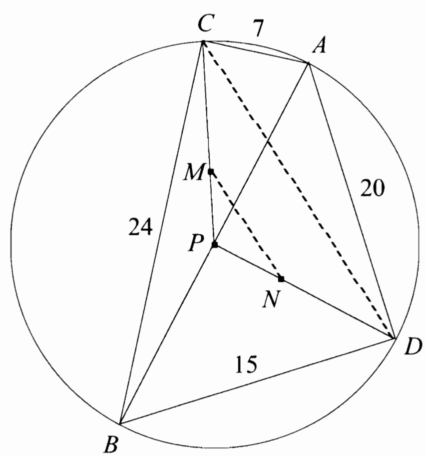
$\dfrac{a(\Delta BAE)}{a(\Delta DCE)} = \dfrac{AB^2}{DC^2} = 64$. Since $\Delta PBC \sim \Delta PDA$, then $\dfrac{a(\Delta PBC)}{a(\Delta PDA)} = \dfrac{PC^2}{PA^2} = 4$. Let $a(PAEC) = K$

and $a(\Delta BAE) = 64N$, making $a(\Delta DCE) = N$. Since $a(\Delta PBC) = a(PAEC) + a(\Delta BAE)$ and

$a(\Delta PDA) = a(PAEC) + a(\Delta DCE)$, then $\dfrac{a(\Delta PBC)}{a(\Delta PDA)} = \dfrac{K + 64N}{K + N} = 4$, making $K = \dfrac{60N}{3}$.

So, $\dfrac{a(PAEC)}{a(\Delta BDE)} = \dfrac{(60N)/3}{64N} = \boxed{\dfrac{5}{16}}$.

T–9. Since $20^2 + 15^2 = 25^2 = 24^2 + 7^2$, both ΔABC and ΔABD are right triangles, so C and D lie on the circle with diameter $\overline{AB}$ and center P. By Ptolemy's Theorem, $(AC)(BD) + (BC)(AD) = (AB)(CD)$, making $CD = \dfrac{7 \cdot 15 + 20 \cdot 24}{25} = \dfrac{117}{5}$. The centroids M and N are the intersection points of the medians and hence, M and N trisect $\overline{CP}$ and $\overline{DP}$ respectively. $\overline{MN} \parallel \overline{CD}$ and since $\dfrac{MP}{PC} = \dfrac{NP}{PD} = \dfrac{1}{3}$, $MN = \dfrac{CD}{3} = \boxed{\dfrac{39}{5}}$.



T–10. The equation $\log_{10} n = a(\log_{10} 14) + b(\log_{10} 15) + c(\log_{10} 16) + d = \log_{10}\left(14^a \cdot 15^b \cdot 16^c \cdot 10^d\right)$ can be rewritten as $n = 14^a \cdot 15^b \cdot 16^c \cdot 10^d = 2^{a+4c+d} \cdot 3^b \cdot 5^{b+d} \cdot 7^a$. So n has no prime factors other than 2, 3, 5, and 7. Conversely, if n has the form $2^p \cdot 3^q \cdot 5^r \cdot 7^s$ for some non-negative integers p, q, r, and s, then we can always find rational a, b, c, d such that $p = a + 4c + d$, $q = b$, $r = b + d$, and $s = a$; i.e., we solve these equations and obtain $a = s, b = q, d = r - q$, and $c = \dfrac{p - s + q - r}{4}$. So the number of logs that can be written as a sum of the given logs is simply the number of elements in $\{1, 2, 3, \ldots, 100\}$ that have no prime factors greater than 7. Compute $100 - \left(\left[\dfrac{100}{11}\right] + \left[\dfrac{100}{13}\right] + \ldots + \left[\dfrac{100}{97}\right]\right)$ where $[x]$ stands for the greatest integer function or write down all positive integers from 1 to 100, count 1 and all those with factors of only 2, 3, 5, 7 to obtain $\boxed{46}$.

ARML Power Question – 1995

Consider a collection of piles of bananas consisting of one pile of 9 bananas, one pile of 6 and one pile of 2. Such a collection, C_1, could be expressed as (9, 6, 2). Obtain a new collection, C_2, by <u>harvesting</u> C_1 where <u>harvesting</u> is defined to mean remove one banana from each pile to form a new pile. Thus, $C_2 = (8, 5, 3, 1)$ and, if C_2 is harvested, we obtain $C_3 = (7, 4, 4, 2)$.

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1. a) Let $C_1 = (8, 5, 2)$. Determine C_2, C_3, C_4, and C_{100}.

   b) Given the collection $M_1 = (7, 6, 5)$, determine M_{1995}. Explain how you arrived at that result.

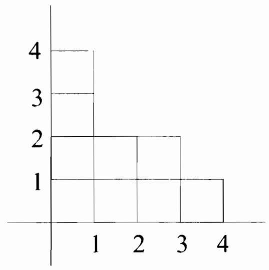
2. Prove that the collection $(k, k-1, k-2, \ldots, 3, 2, 1)$ remains fixed after harvesting.

3. Prove that any collection of piles whose total number of bananas is 6 can be reduced to the collection (3, 2, 1) by successive harvesting. Determine the maximum number of harvests required.

4. Denote by $C(n_k, n_{k-1}, \ldots, n_2, n_1)$ a collection of *k* piles of size n_k through n_1 ordered such that $n_k \geq n_{k-1} \geq \ldots \geq n_1$. Suppose $n_k + n_{k-1} + \ldots + n_1 = 7$. Determine the number of collections *C* such that the 1995th harvest yields the collection (4, 2, 1). Justify your answer.

5. Let $C = (n)$, that is, let collection *C* be one pile of *n* bananas. If *C* is harvested successively, the collections will eventually fall into a periodic sequence. That is, for all *k* sufficiently large, $C_k = C_{k+p}$. If the harvest is fixed after some point, then $p = 1$. Determine all values of *n* such that $p = 1995$. Explain your reasoning; you need not prove your result.

Define the graph of a collection as follows: write the collection of piles in non-increasing order from left to right, then draw a column of squares for each pile. The graph of (4, 2, 2, 1) is shown at the right:



Let the level of a square denote the sum of the coordinates of its center. Let the level sum of a collection be the sum of all the levels of the squares of the collection.

6. Does there exist a collection of piles different from $(n, n-1, \ldots, 3, 2, 1)$ such that after harvesting once, the level sum remains the same?

7. Find the level sum of $C = (n, n-1, n-2, \ldots, 3, 2, 1)$.

8. Show that the level sum of any collection does not increase following a harvest.

9. Prove that if a collection of piles contains $\frac{n(n+1)}{2}$ bananas for $n \geq 2$, then successive harvesting will eventually lead to the collection $(n, n-1, \ldots, 2, 1)$.

10. Prove that a collection containing one pile of $\frac{n(n+1)}{2}$ bananas will be reduced to the collection $(n, n-1, \ldots, 3, 2, 1)$ in exactly $\frac{n(n-1)}{2}$ harvests.

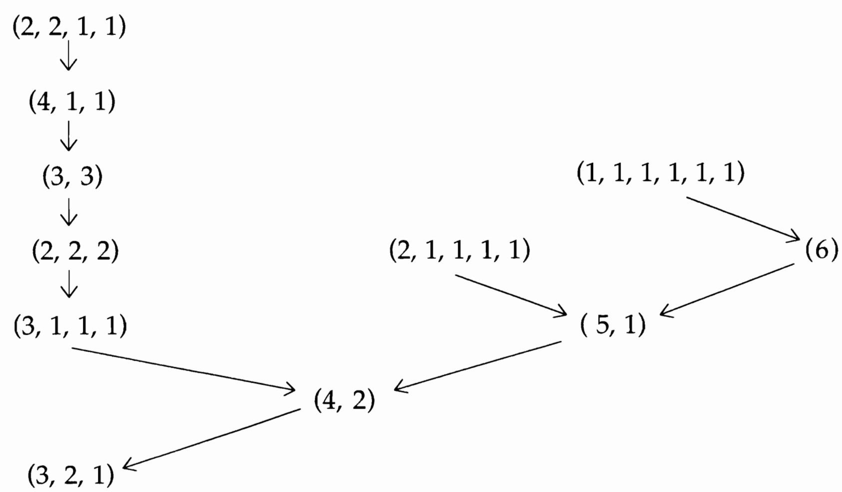
Solutions to the ARML Power Question – 1995

1a. $C_1(8,5,2) \to C_2(7,4,3,1) \to C_3(6,4,3,2) \to C_4(5,4,3,2,1) \to C_5(5,4,3,2,1) \to C_{100}(5,4,3,2,1)$. The harvest is fixed or repeats with a period of 1.

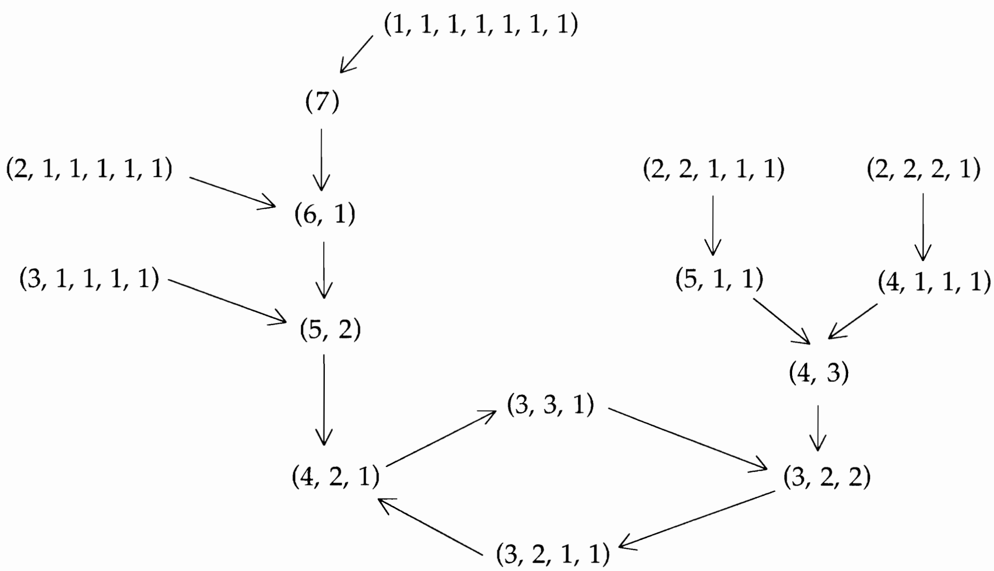
1b. $M_1(7,6,5) \to M_2(6,5,4,3) \to M_3(5,4,4,3,2) \to M_4(5,4,3,3,2,1) \to M_5(6,5,3,2,2,1) \to M_6(6,5,3,2,1,1) \to M_7(6,5,4,2,1) \to M_8(5,5,4,3,1) \to M_9(5,4,4,3,2) \to M_{10}(5,4,3,3,2,1)$. Since $M_9 = M_3$, harvesting (7, 6, 5) is periodic with period of 6 beginning with M_3, making $M_{3+6k} = M_3$. Since $1995 = 3 + 6(332)$, $M_{1995} = M_9 = (5, 4, 4, 3, 2)$.

2. To harvest $(k, k-1, \ldots, 3, 2, 1)$, we reduce each pile by 1, obtaining piles of size $k-1, k-2, \ldots, 2, 1, 0$, and one new pile whose size equals the number of piles, namely k. This yields the original collection $(k, k-1, \ldots, 3, 2, 1)$. Thus, the process terminates whenever $(k, k-1, \ldots, 3, 2, 1)$ is reached.

3. The following diagram represents all collections of piles with 6 bananas and the harvest of each. The maximum number of harvests required to reach (3, 2, 1) is 6.



4. There are 15 collections with 7 bananas. If harvested successively all yield (4, 2, 1). Since (4, 2, 1) $\rightarrow$ (3, 3, 1) $\rightarrow$ (3, 2, 2) $\rightarrow$ (3, 2, 1, 1) $\rightarrow$ (4, 2, 1), harvesting has a period of 4 once (4, 2, 1) has been reached. If a collection yields (4, 2, 1) on the $3 + 4k$ harvest, then it will yield (4, 2, 1) on the 1995th harvest. There are 4 collections with that property: (7), (3, 3, 1), (4, 3), (2, 1, 1, 1, 1, 1). The following diagram shows all collections of piles with 7 bananas and the harvest of each:



5. As shown in #3 and #4, the period (6) yields is 1, the period (7) yields is 4. We form a table showing n and p:

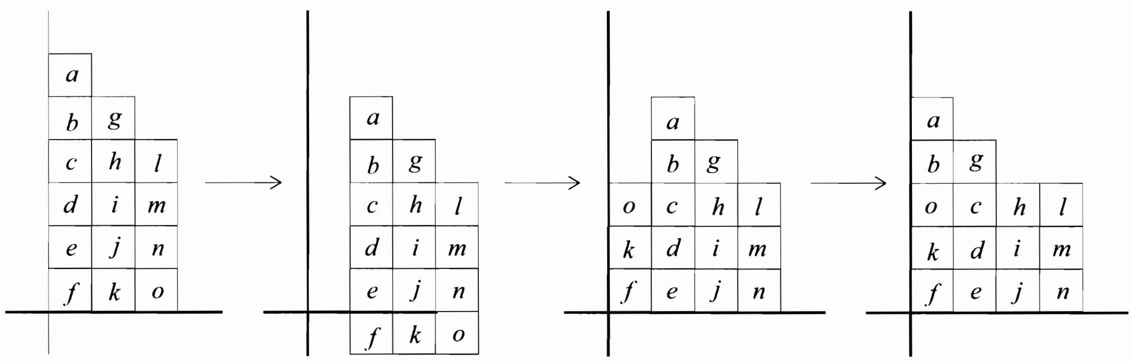
n	1	2	3	4	5	6	7	8	9	10	11	12	13
p	1	2	1	3	3	1	4	4	4	1	5	5	5

When $n = 1, 3, 6, 10, \ldots \dfrac{k(k+1)}{2}$, the period is 1. For $n = \dfrac{k(k+1)}{2} + m$ for $0 < m \leq k$, the period appears to be $k + 1$. If the period is 1995, then $k = 1994$ and n will range from $\dfrac{1994 \cdot 1995}{2} + 1$ to $\dfrac{1994 \cdot 1995}{2} + 1994$, that is, from 1,989,016 to 1,991,009 inclusive.

6. Consider harvesting a collection of 1 pile of two bananas: $(2) \to (1, 1)$. The level sum for (2) is $\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{3}{2}\right) = 3$ and the level sum for (1, 1) is $\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{3}{2}+\frac{1}{2}\right) = 3$.

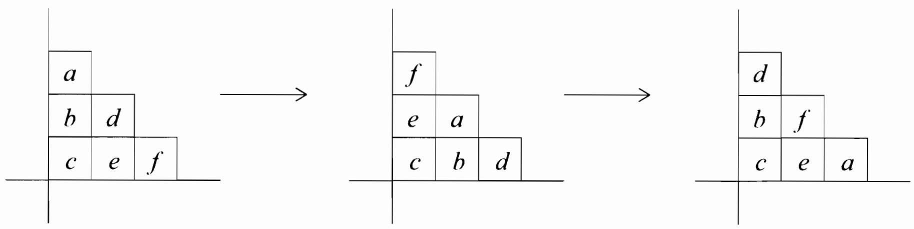
7. Clearly all the squares in the graph of C are of a level less than or equal to n, and for each $k = 1, 2, \ldots, n$ the graph contains exactly k squares of level k. Thus the level sum of $C = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

8. Consider a harvest without reordering, i.e., let the pile corresponding to the number of piles be listed first. Thus $(6, 5, 4) \to (3, 5, 4, 3)$.

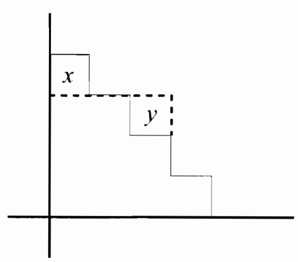


The center of each square is shifted one to the right and one down. Thus the sum of the coordinates of the centers of the squares above the x-axis remains constant. Suppose there were r piles before harvesting. Before harvesting, the sum of the x-coordinates of the centers of the first row of squares equals $\frac{1}{2}+\frac{3}{2}+\ldots+\frac{2r-1}{2} = \frac{r^2}{2}$. The sum of the corresponding y-coordinates equals $r \cdot \frac{1}{2}$. If the first row becomes the first column, then the sum of the x-values equals $r \cdot \frac{1}{2}$ and the sum of the y-values equals $\frac{1}{2}+\frac{3}{2}+\ldots+\frac{2r-1}{2} = \frac{r^2}{2}$. Thus, the level sum is unchanged. If the first column contains at least as many squares as any other column, the harvest is properly ordered and the level sum is the sum as before harvesting. If the first column contains less than the second or third, etc., as in the diagram above, then order the collection by shifting squares to the left. In this, case squares $\boxed{a}$ and $\boxed{b}$ are shifted to column 1 and square $\boxed{g}$ is shifted to column 2. A shift to the left reduces each x-coordinate of the center by 1, thereby decreasing the level sum. Thus, the level sum is constant or decreases.

9. Let $S = (n, n-1, \ldots, 2, 1)$. Suppose that there exist collections containing $\frac{n(n+1)}{2}$ bananas that do not lead to S by successive harvesting. Let C be such a collection with the additional property that it has the minimal level sum of all such collections. Since C does not lead to S, no collection derived from C by harvesting does either. Additionally, the level sum of C cannot decrease by harvesting since C is supposed to have minimal level sum. Consider the proof of #8: the only case in which the level sum of C remains constant under harvesting is that in which the first harvested column contains as many squares as any other column. In that case, the individual labeled squares move either one square SE or the maximum number of squares NW as when, for example, the center of d moves from $\left(\frac{3}{2}, \frac{3}{2}\right)$ to $\left(\frac{5}{2}, \frac{1}{2}\right)$ to $\left(\frac{1}{2}, \frac{5}{2}\right)$. Under this movement, the squares cycle with period equal to their respective level; they move along diagonals and since the level along a diagonal is constant, the level of each square remains constant.



With these observations in mind, we derive a contradiction. Since $C \neq (n, n-1, \ldots, 2, 1)$, C is not equal to the set of squares with level less than or equal to n. But the number of elements in C equals $\frac{n(n+1)}{2}$ so C must have an element y not included in S and S must have an element x not included in C. Intuitively, y is an extra square and x is a hole. Shown is $C = (3, 3, 3, 1)$ and $S = (4, 3, 2, 1)$.



If the level of y is greater than $n + 1$, there is some square y' below y in C with level exactly $n + 1$. But y' is not an element of S because the level of y' is greater than n. Similarly, if the level of x is less than n, there is a square x' of level n above x also not in C but in S because x' has level n. By the logic of #8, under successive harvesting y' rotates with period $n + 1$, and x' rotates with period n. After $n + 1$ harvests, x' moves one square SE, y' stays where it is. Therefore, after at most $n(n + 1)$ harvests, y' is directly above x'. This means a square of C is directly above a hole of C. This is a contradiction.

10. We prove by induction that after $j = \dfrac{k(k+1)}{2} + m$ harvests where $m \leq k$, our collection consists of:

- one pile of $\dfrac{n(n+1)}{2} - j$ bananas
- m piles of $k+1, k, \ldots, k-m+2$ bananas
- $k-m$ piles of $k-m, \ldots, 1$ bananas.

The base case, $j = 0$, is obvious, since if $j = 0$, then $k = m = 0$, yielding one pile of $\dfrac{n(n+1)}{2}$ bananas.

We divide the inductive step into three cases: i) $m = 0$, ii) $0 < m < k$, iii) $m = k$.

i) If $m = 0$, we begin with 1 pile of $\dfrac{n(n+1)}{2} - j$ bananas and k piles of $k, k-1, \ldots, 1$ bananas.

Harvesting yields 1 pile of $\dfrac{n(n+1)}{2} - j - 1$, $k-1$ piles of $k-1, k-2, \ldots, 1$, and 1 pile of $k+1$ bananas.

Rewritten as 1 pile of $\dfrac{n(n+1)}{2} - (j+1)$, $k-1$ piles of $k-1, \ldots, 1$, and 1 pile of $k+1$, we see that $m = 1$, k is unchanged and j is increased by 1.

ii) If $0 < m < k$, harvesting yields 1 pile of $\dfrac{n(n+1)}{2} - j - 1$ bananas, m piles of $k, \ldots, k-m+1$ bananas, $k-m-1$ piles of $k-m-1, \ldots, 1$ bananas and 1 pile consisting of $1 + m + k - m = k+1$ bananas. This gives $m+1$ piles of $k+1, k, \ldots, k-m+1$ bananas. This is exactly as expected, j and m both being increased by 1.

iii) If $m = k$, harvesting yields 1 pile of $\dfrac{n(n+1)}{2} - j - 1$ bananas and $m+1$ piles of $k+1, \ldots, 1$ bananas. Again, j and m are increased by 1.

After $\dfrac{n(n-1)}{2}$ harvests, $k = n-1$ and $m = 0$. This results in 1 pile of $\dfrac{n(n+1)}{2} - \dfrac{n(n-1)}{2} = n$ bananas, zero piles of $k+1, \ldots, k-m+2$ bananas, and $n-1$ piles of $n-1, \ldots, 1$. This yields the desired collection: $(n, n-1, \ldots, 1)$.

I–1. Compute the largest prime factor of:

$$3\left(3\left(3\left(3\left(3\left(3\left(3\left(3\left(3\left(3(3+1)+1\right)+1\right)+1\right)+1\right)+1\right)+1\right)+1\right)+1\right)+1\right)+1$$

I–2. Triangle ABC is similar to ΔMNP. If $BC = 60$, $AB = 12$, $MP = 8$, and $BC > AC > AB$, compute the sum of all possible integer values for the ratio of the area of ΔABC to the area of ΔMNP.

I–3. Determine all positive primes p such that $p^{1994} + p^{1995}$ is a perfect square.

I–4. Let $.\overline{A} = .AAAA\ldots$. Compute the number of distinct ordered triples (A, B, C) with $A, B, C \in \{0, 1, 2, \ldots, 9\}$ such that k is an integer if

$$k = \frac{.\overline{ABC} + .\overline{ACB} + .\overline{BAC} + .\overline{BCA} + .\overline{CAB} + .\overline{CBA}}{.\overline{A} + .\overline{B} + .\overline{C}}.$$

I–5. Compute the number of distinct planes passing through at least three vertices of a given cube.

I–6. Determine all positive integers $k \leq 2000$ for which $x^4 + k$ can be factored into two distinct trinomial factors with integer coefficients.

I–7. Let $f(x) = \sqrt{x+2} + c$. Determine all real values of c such that the graphs of $f(x)$ and its inverse $f^{-1}(x)$ intersect in two distinct points.

I–8. For x and y in radians, compute the number of solutions in ordered pairs (x, y) to the following system:

$$\sin(x+y) = \cos(x+y)$$

$$x^2 + y^2 = \left(\frac{1995\pi}{4}\right)^2$$

1. 73
2. 930
3. 3
4. 999
5. 20
6. 4, 64, 324, 1024
7. $-2.25 < c \leq -2$ or its equivalent $(-2.25, -2]$
8. 3990

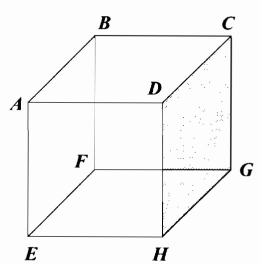
I–1. The given expression equals $3^{11} + 3^{10} + 3^9 + \ldots + 3^1 + 1 = \dfrac{3^{12} - 1}{3 - 1}$. The numerator factors as $\left(3^6 + 1\right)\left(3^6 - 1\right) = \left(3^2 + 1\right)\left(3^4 - 3^2 + 1\right)\left(3^3 - 1\right)\left(3^3 + 1\right) = 10 \cdot 73 \cdot 26 \cdot 28$. The largest prime factor is $\boxed{73}$.

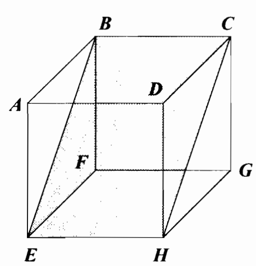
I–2. Clearly $12 < AC < 60$. By the triangle inequality, $60 - 12 < AC < 60 + 12 \rightarrow$ $48 < AC < 60$. Thus, $\left(\dfrac{48}{8}\right)^2 < \dfrac{\text{area of } \Delta ABC}{\text{area of } \Delta MNP} < \left(\dfrac{60}{8}\right)^2 \rightarrow 36 < \dfrac{\text{area of } \Delta ABC}{\text{area of } \Delta MNP} < 56.25$. Thus, the ratio of areas lies in the set $\{37, 38, \ldots, 56\}$. The sum of all such ratios equals $\dfrac{(37 + 56)20}{2} = \boxed{930}$.

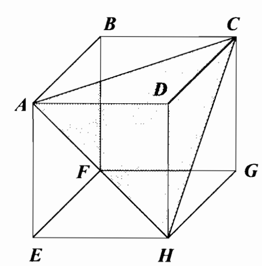
I–3. In the general case, since $p^{2n} + p^{2n+1} = p^{2n}(1 + p) = k^2$ for k an integer, clearly $1 + p$ must be a perfect square. Setting $1 + p = t^2 \rightarrow p = t^2 - 1 = (t + 1)(t - 1)$. This is prime if and only if $t - 1 = 1$, making $t = 2$. Thus $p = \boxed{3}$.

I–4. Since $.\overline{ABC} = \dfrac{100A + 10B + C}{999}$ and A, B, and C are each in the hundreds, tens, and ones place twice in the six decimals, the numerator equals $\dfrac{222A + 222B + 222C}{999} = \dfrac{2A + 2B + 2C}{9}$. Since $.\overline{A} = \dfrac{A}{9}$, the denominator equals $\dfrac{A + B + C}{9}$. Thus, the quotient surprisingly equals 2 for all triples of A, B, and C except (0, 0, 0). The number of triples equals $(10)(10)(10) - 1 = \boxed{999}$.

I–5. The planes are of three types: 1) <u>Face planes</u> such as *DCGH*, 1 for each face for a total of 6.
2) <u>Opposite edge planes</u> such as *BCHE*, 1 for each pair of opposite edges for a total of 12/2 = 6.
3) <u>Face diagonal planes</u> such as *ACH* forming the base of pyramid *D-ACH*. Each corner can serve as a vertex of such a pyramid, so there are 8 such planes. Total: $6 + 6 + 8 = \boxed{20}$.

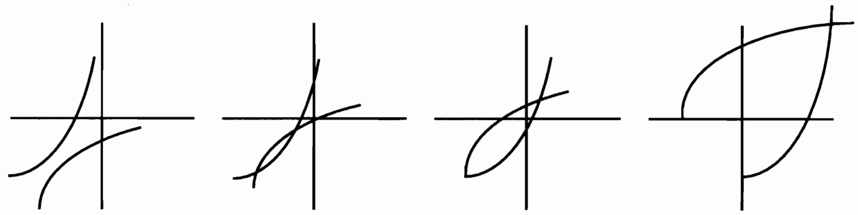






I–6. Note that $x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = \left(x^2 + 2\right)^2 - (2x)^2$. The latter expression factors as $\left(x^2 - 2x + 2\right)\left(x^2 + 2x + 2\right)$. Hence, set $x^4 + k = \left(x^2 + n\right)^2 - 2nx^2 = \left(x^2 + n\right)^2 - \left(x\sqrt{2n}\right)^2 = \left(x^2 - x\sqrt{2n} + n\right)\left(x^2 + x\sqrt{2n} + n\right)$. Since $\sqrt{2n}$ must be an integer and $n^2 = k$, then $\sqrt{2\sqrt{k}}$ must be integral. Let $k = 4m$. Then $2\sqrt[4]{m}$ must be integral. Thus, $m = 1^4, 2^4, 3^4, 4^4, 5^4, \ldots$ making $k = 4, 64, 324, 1024, 2500, \ldots$ Choose $\boxed{4, 64, 324, 1024}$.

I–7. $f^{-1}(x) = (x - c)^2 - 2$. The graphs of f and f^{-1} are monotonically increasing and are symmetric over $y = x$. Thus, they intersect on $y = x$ and the number of intersections equals the number of solutions to $f(x) = x$. Set $(x - c)^2 - 2 = x \to x^2 - (2c + 1)x + c^2 - 2 = 0$. There are two solutions if the discriminant is positive. Thus, $(2c + 1)^2 - 4\left(c^2 - 2\right) > 0 \;\to\; c > -2.25$. As c increases the number of solutions goes from 0 to 1 to 2 to 1. The transition from 2 to 1 occurs when $c = -2$. Thus, $\boxed{-2.25 < c \le -2}$.



I–8. $\sin(x + y) - \cos(x - y) = 0 \to (\sin x \cdot \cos y + \cos x \cdot \sin y) - (\cos x \cdot \cos y + \sin x \cdot \sin y) = 0 \to (\sin x)(\cos y - \sin y) - (\cos x)(\cos y - \sin y) = 0 \to (\sin x - \cos x)(\cos y - \sin y) = 0 \to \sin x = \cos x$ or $\sin y = \cos y$. Therefore, $x = \frac{\pi}{4} + n\pi$ or $y = \frac{\pi}{4} + n\pi$ for n an integer. The radius of the circle is $\frac{1995\pi}{4}$.

For $x \ge 0$, there are vertical lines at $x = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots, \frac{1993\pi}{4}$ for a total of 499 vertical lines intersecting the circle at two points, making 998 points of intersection. Similarly for $y \ge 0$ there are 499 horizontal lines intersecting the circle at 998 points. For $x < 0$ there are vertical lines at $x = -\frac{3\pi}{4}, -\frac{7\pi}{4}, \ldots, -\frac{1995\pi}{4}$.

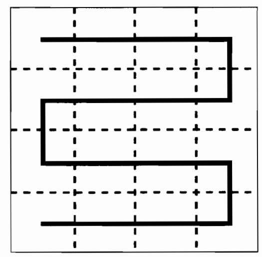
There are $\frac{1995 - 3}{4} + 1 = 499$ vertical lines, 498 of which intersect the circle in 2 points, the other is tangent. Thus, there are 997 points of intersection. Similarly, for $y < 0$ there are 997 points of intersection. The total number of points of intersection equals $2(998) + 2(997) = \boxed{3990}$.

ARML Relay #1 – 1995

R1–1. Compute the number of distinct 11 letter palindromic permutations of the letters of MISSISSIPPI.

R1–2. Let K = TNYWR. If the regular octagon $ABCDEFGH$ has sides equal to $\frac{K}{30}$ in length, compute the value of $AC^2 - AD$.

R1–3. Let M = TNYWR. Let S_n be a unit square subdivided into 2^{2n} congruent squares. The centers are all connected as shown in the diagram at the right for S_2. Compute the least value of n such that the sum of the lengths of the connecting line segments exceeds 10^{4M}.



ARML Relay # 2 – 1995

R2–1. Let $ABCDEF$ be a regular hexagon. If the ratio of the area of region $ABCE$ to the area of the hexagon in simplest terms is $\frac{a}{b}$, compute $a + b$.

R2–2. Let K = TNYWR. Compute the number of sets of K consecutive integers in $\{1, 2, 3, \ldots, 50\}$ such that the product of the elements of each set of K integers is divisible by both 12 and 21.

R2–3. Let M = TNYWR. Given points $D(0, M)$, $O(0, 0)$, and $E(M, 0)$, equilateral triangle ABC is inscribed in ΔDOE such that C is the midpoint of $\overline{DE}$ and $\overline{AB}$ is parallel to $\overline{DE}$. The length of $\overline{AB}$ can be expressed in simplest form as $a\left(\sqrt{b} - \sqrt{c}\right)$. Determine (a, b, c).

Relay #1:

R1–1. 30

R1–2. 1

R1–3. 14

Relay #2:

R2–1. 5

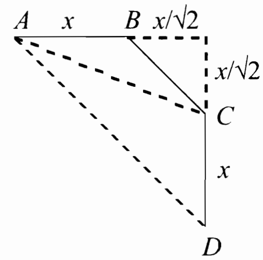
R2–2. 32

R2–3. (16, 6, 2)

Solutions to ARML Relay #1 – 1995

R1–1. M must lie in the 6th position. To its left are permutations of $SSIPP$, to its right are mirror images of letters on its left. Thus, M and letters to its right are fixed. The number of permutations is $\dfrac{5!}{2!\cdot 2!\cdot 1!} = \boxed{30}$.

R1–2. Let $\dfrac{K}{30} = x$. Then $AC^2 = \left(x + \dfrac{x}{\sqrt{2}}\right)^2 + \left(\dfrac{x}{\sqrt{2}}\right)^2 = x^2\left(2 + \sqrt{2}\right)$.



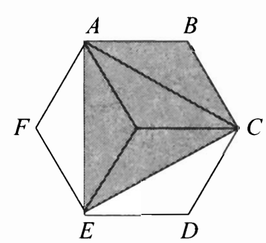
Also, $AD = \sqrt{2}\left(x + \dfrac{x}{\sqrt{2}}\right) = x\left(\sqrt{2} + 1\right)$. Since $K = 30$, then $x = 1$ so

$AC^2 - AD = \left(2 + \sqrt{2}\right) - \left(\sqrt{2} + 1\right) = \boxed{1}$.

R1–3. There are 2^n horizontal line segments of length $1 - \dfrac{1}{2^n}$. The sum of the vertical line segments equals a horizontal line segment which equals $1 - \dfrac{1}{2^n}$. The sum of the vertical and horizontal lengths equals $\left(2^n + 1\right)\left(1 - \dfrac{1}{2^n}\right) = 2^n - \dfrac{1}{2^n} \approx 2^n$. Since $M = 1$ then we want to find n such that $2^n > 10^4$. We have $2^{10} = 1024$, so $2^{13} = 8192$ and $2^{14} = 16384$, so $n = \boxed{14}$.

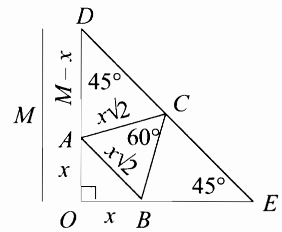
Solutions to ARML Relay #2 – 1995

R2–1. From the diagram, $\dfrac{\text{area of } ABCE}{\text{area of } ABCDEF} = \dfrac{4}{6} = \dfrac{2}{3}$. The sum equals $2 + 3 = \boxed{5}$.



R2–2. Since $K = 5$, we seek the number of sets of five consecutive integers in $S = \{1, 2, \ldots, 50\}$ such that the product of the elements of each 5-tuple is divisible by $2^2 \cdot 3 \cdot 7$. Any five consecutive numbers is divisible by 4 and 3 so each of the desired 5-tuples must contain a multiple of 7. For each multiple of 7 from 7 to 42 there are five 5-tuples in S containing the multiple of 7 in S and for 49 there are two such 5-tuples in S. Thus, there are $6 \cdot 5 + 2 = \boxed{32}$ sets of five integers the product of whose elements is divisible by both 12 and 21.

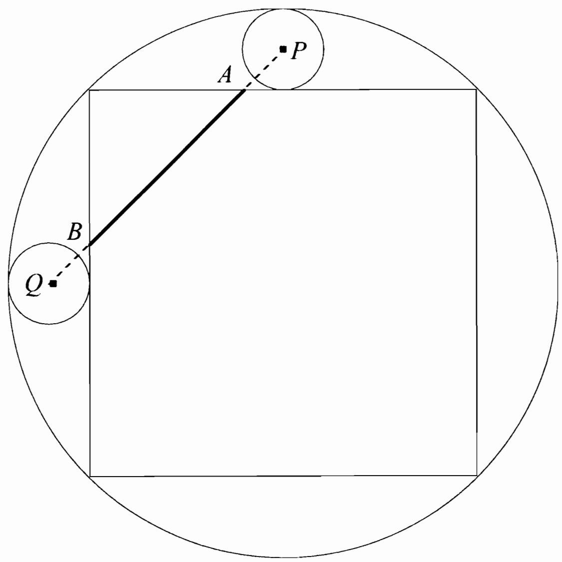
R2–3. Let $OA = OB = x$ giving $AB = AC = x\sqrt{2}$. Draw $\overline{OC} \perp \overline{DE}$. Since $m\angle ACD = 60°$, then by the Law of Sines, $\dfrac{AC}{\sin 45°} = \dfrac{AD}{\sin 60°} \rightarrow$ $AC = \dfrac{AD \sin 45°}{\sin 60°} = \dfrac{AD\sqrt{2}}{\sqrt{3}}$. Since $AD = M - x$, then



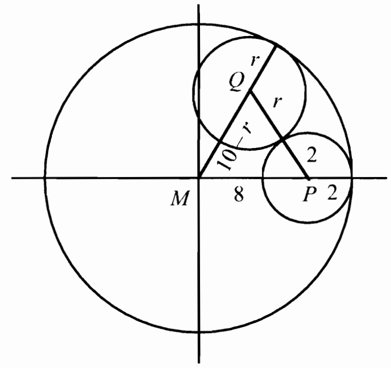
$AC = \dfrac{(M - x)\sqrt{2}}{\sqrt{3}} = x\sqrt{2}$. Solving for x, we obtain $x = \dfrac{M(\sqrt{3} - 1)}{2}$ making $AC = x\sqrt{2} = \dfrac{M(\sqrt{6} - \sqrt{2})}{2}$. Since $M = 32$, $AC = 16(\sqrt{6} - \sqrt{2})$. Since $AC = AB$, the answer is $\boxed{(16, 6, 2)}$.

1. The center and radius of circle M are $M(0, 0)$ and 10 respectively. The center and radius of circle P are $P(8, 0)$ and 2 respectively. A circle with center Q is internally tangent to circle M and externally tangent to circle P. Compute the maximum y-value of the locus of points Q.

2. A square is inscribed in a circle of radius 1. Circles P and Q are the largest circles that can be inscribed in the indicated segments of the circle. The line joining the centers of circles P and Q intersects the square in points A and B. Compute the length of $\overline{AB}$.



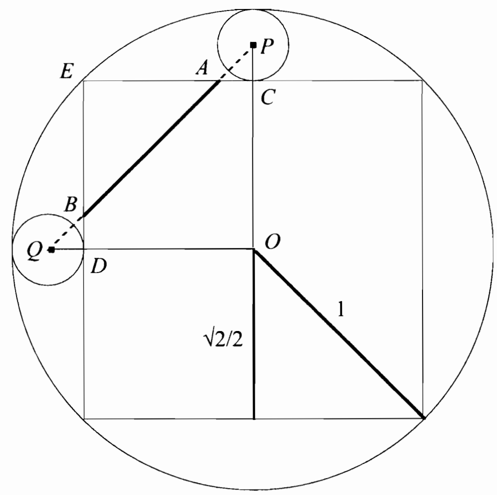
1. Since the radius of circle $Q = 10 - MQ = PQ - 2$, then $MQ + PQ = 12$. This implies that the locus of Q is an ellipse with focal points M and P. Thus, the equation of the locus of Q is $\frac{(x-4)^2}{36} + \frac{y^2}{20} = 1$. If $x = 4$, then $y = \sqrt{20} = \boxed{2\sqrt{5}}$.



2. Half the side of the square is $\frac{\sqrt{2}}{2}$. Thus,

$$CP = DQ = \frac{1 - \frac{\sqrt{2}}{2}}{2} = \frac{2-\sqrt{2}}{4} = AC = BD.$$ This gives $AE = EC - AC = \frac{\sqrt{2}}{2} - \frac{2-\sqrt{2}}{4} = \frac{3\sqrt{2}-2}{4}$.

Since $AB = \sqrt{2} \cdot AE$, then $AB = \boxed{\frac{3-\sqrt{2}}{2}}$.



ARML

1996

The 21st ANNUAL MEET

ARML celebrated its twenty-first year of competition with its largest competition ever. The second year of the University of Nevada at Las Vegas site saw an increased presence of strong California teams and the first western winner ever. New teams from Washington and California Davis/Sacramento joined the ranks. The former Northern California teams were replaced by three teams from the San Francisco Bay area and the Metro New York teams were replaced by the Westchester, Suffolk, and Nassau teams. We were again honored with the presence of a team from Russia. Including eight alternate teams, a total of eight-nine teams and over 1335 students participated.

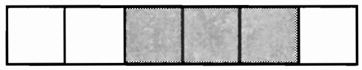
Kay Tipton and Dorothy Wendt of the Alabama team received the Samuel Greitzer Distinguished Coach Award. They established the team in 1988 and have finished in the top 15 every year except 1992 when the team couldn't make the trip.

Barbara Rockow received the Alfred Kalfus Founder's Award. Barbara has been a long time ARML supporter and has been the corresponding secretary as well as a force behind the scenes these past 12 years.

Michael Colsher of Wisconsin received the Zachary Sobol Award given for outstanding contributions to his ARML team. Michael helped organize and train his team.

T–1. A cylindrical container 10 units high and 4 units in diameter is partially filled with water. The cylinder is tilted so that the water level reaches 9 units up the side of the cylinder at the highest but only 3 units up at the lowest. Compute the volume of water in the cylinder.

T–2. Imagine 173 unit squares arranged in a row, 6 of which are shown. If a rectangle consists of a single square, or a combination of consecutive squares as indicated by the shaded region, compute the total number of rectangles that can be formed.



T–3. Let $Q(x)$ be the quotient when $37x^{73} - 73x^{37} + 36$ is divided by $x - 1$. Compute the sum of the coefficients of $Q(x)$.

T–4. A positive integer N is self-descriptive if each digit of N appears as often as the value of the digit. For example, 212 is a three-digit self-descriptive number. Compute the sum of the digits of all the six-digit self-descriptive numbers.

T–5. Let $M = 1\ 1\ .\ .\ .\ .\ .\ 1\ 2$ where M is a base 10 integer consisting of 299 1's followed by a 2 in the unit's place. Let N be a positive integer. If the product $(M \cdot N)$ has K times as many digits as N, compute the largest possible value for K.

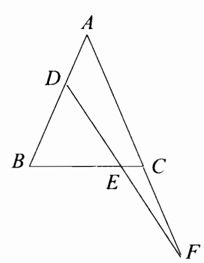
T–6. Given two concentric circles with center O, points A and B lie on the inner circle, point P lies on the outer circle. $\overline{PA}$ and $\overline{PB}$ are tangent to the inner circle at A and B respectively. If $\tan\angle AOB - \tan\angle APB = -8/3$, the ratio of the radius of the large circle to the radius of the small circle can be written as $\sqrt{x}$ where x is an integer. Compute x.

T–7. An icosahedron is a regular polyhedron whose faces are 20 congruent equilateral triangles. Moving on the edges of an icosahedron, compute the number of shortest paths from a given vertex A to the vertex opposite A.

T–8. Distinct lines L_1 and L_2 with positive slopes m_1 and m_2 respectively, pass through $P(3,10)$. The area of the triangle formed by P and the x-intercepts of L_1 and L_2 equals 200. If $m_1 \le .24$, compute the number of integer values m_2 can take on.

T–9. In ΔABC, $AB = AC = 115$, $AD = 38$, and $CF = 77$. In simplest form the ratio of the area of ΔCEF to the area of ΔDBE can be written as $\dfrac{m}{n}$.

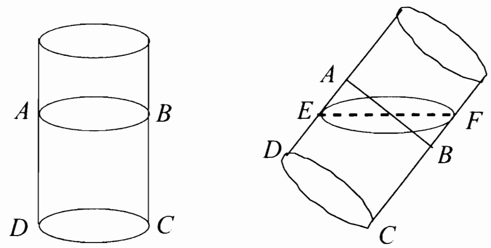
Compute the ordered pair (m, n).



T–10. Let $N(x) = \left[\left[\left[\left[[x] + 6x\right] + 15x\right] + 65x\right] + 143x\right]$ where the brackets represent the greatest integer function. If the domain of N is $\{x: 0 \le x \le 1\}$, compute the number of values in the range of N.

1. 24π
2. 15051
3. 0
4. 1332
5. 301
6. 5
7. 10
8. 6
9. (19, 96)
10. 209

T–1. Since $DE = 3$, $FC = 9$ and $AE = FB = x$, then $3 + x = 9 - x$, making $x = 3$, so that the height AD of the water is 6. Thus, the volume = $\pi\left(2^2\right) \cdot 6 = \boxed{24\pi}$.



2nd Method: Imagine that the water in the tilted cylinder is frozen. Form a congruent solid and join the two to form a cylinder of diameter 4 and height 12. Its volume is 48π, so half of 48π is the answer.

T–2. A rectangle is formed when any 2 of the 174 vertical segments are chosen. Thus, ${}_{174}C_2 = \boxed{15051}$.

T–3. Let $P(x) = 37x^{73} - 73x^{37} + 36 = 37\left(x^{73} - 1\right) - 73\left(x^{37} - 1\right)$ which equals $37(x-1)\left(x^{72} + x^{71} + \ldots + 1\right) - 73(x-1)\left(x^{36} + x^{35} + \ldots + 1\right)$. Thus, $Q(x)$ equals $37\left(x^{72} + \ldots + 1\right) - 73\left(x^{36} + \ldots + 1\right)$. The sum of the coefficients of $Q(x)$ is $Q(1)$ which equals $37(73) - 73(37) = \boxed{0}$. Note: The original problem involved $19x^{96} - 96x^{19} + 77$, but we chose to suppress our calendrical instincts.

T–4. There is 1 self-descriptive number with six 6's giving a sum of 36. There are $\dfrac{6!}{5!\cdot 1!} = 6$ numbers consisting of five 5's and one 1 giving a sum of 6(26) = 156. There are $\dfrac{6!}{4!\cdot 2!} = 15$ numbers with four 4's and two 2's giving a sum of 15(20) = 300, and there are $\dfrac{6!}{3!\cdot 2!\cdot 1!} = 60$ numbers with three 3's, two 2's, and one 1, giving a total of 60(14) = 840. Answer: $36 + 156 + 300 + 840 = \boxed{1332}$.

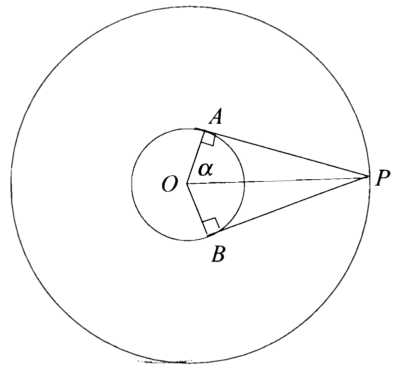
T–5. Suppose N has n digits, making $N < 10^n$. Write $M = \dfrac{10^{300} + 8}{9}$. If MN has Kn digits, then $10^{Kn-1} \le NM$. Thus $9 \cdot 10^{Kn-1} \le \left(10^{300} + 8\right) \cdot N < \left(10^{300} + 8\right) \cdot 10^n = 10^{300+n} + 8 \cdot 10^n$. It follows that $Kn - 1 < 300 + n$, making $K < \dfrac{301}{n} + 1$. K is clearly maximized when $n = 1$ and $N = 9$, making $K = \boxed{301}$.

T–6. Let $m\angle AOP = \alpha$, then $\tan 2\alpha - \tan(180° - 2\alpha) = -\frac{8}{3}$,

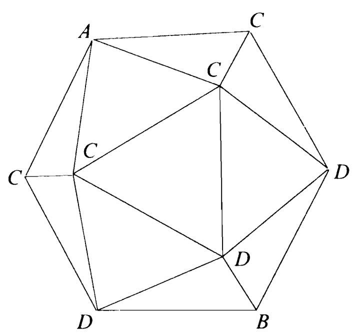
giving $2\tan 2\alpha = -\frac{8}{3} \rightarrow \frac{2\tan\alpha}{1-\tan^2\alpha} = -\frac{4}{3} \rightarrow$

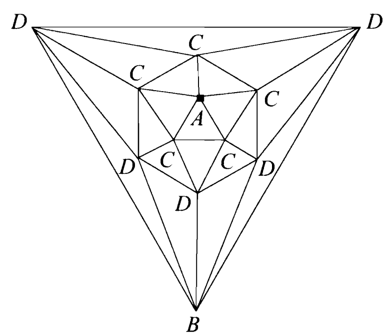
$2\tan^2\alpha - 3\tan\alpha - 2 = 0 \rightarrow (2\tan\alpha + 1)(\tan\alpha - 2) = 0$.

Choose $\tan\alpha = 2$. Without loss of generality, let $AP = 2$, $OA = 1$, making $OP = \sqrt{5}$. The ratio of OP to $OA = \sqrt{5}$. Ans: $\boxed{5}$.



T–7. Consider the icosahedron labeled as indicated at the left below. There is 1 way to get from A to one of the 5 vertices C which are one edge-length away from A. There are two ways to get from each C to each vertex D, one edge-length away . From each vertex D there is 1 way to get to B. Thus the number of shortest paths along the edges from A to B equals $5 \cdot 2 = \boxed{10}$. Shown to the right is a different way to visualize an icosahedron that could be used to solve the problem.





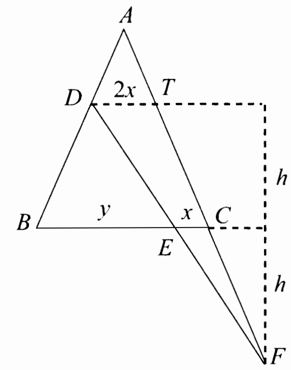
T–8. In general, $y - 10 = m(x-3)$ implies an x-intercept of $3 - \frac{10}{m}$. The base of the triangle equals

$\left|3 - \frac{10}{m_2} - \left(3 - \frac{3}{m_1}\right)\right| = 10\left|\frac{1}{m_1} - \frac{1}{m_2}\right|$. Thus, 200 equals $\frac{1}{2} \cdot 10 \cdot 10\left|\frac{1}{m_1} - \frac{1}{m_2}\right|$ giving

$4 = \left|\frac{m_2 - m_1}{m_1 m_2}\right|$. Since m_2 is an integer and $m_2 > m_1$, we obtain $4 = \frac{m_2 - m_1}{m_1 m_2}$. Solving for m_1,

we obtain $\frac{m_2}{4m_2 + 1} = m_1$. Since $m_1 \le 0.24$, solving for m_2 yields $m_2 \le 6$. Hence $m_2 = 1, 2, \ldots, 6$.

The answer is $\boxed{6}$.



T–9. Draw $\overline{DT}$ parallel to $\overline{BC}$, let $EC = x$, $BE = y$. Since $DB = TC = CF$, then $DT = 2x$. Since $\frac{AD}{DT} = \frac{AB}{BC}$ we obtain $\frac{38}{2x} = \frac{115}{x+y}$. Solve for y and obtain $y = \frac{96x}{19}$. Since $\overline{BC}$ bisects the altitude from F to $\overline{DT}$, the altitude from D to $\overline{BE}$ equals the altitude from F to $\overline{EC}$, consequently the ratio of the area of ΔCEF to the area of ΔDBE equals $\frac{EC}{BE} = \frac{x}{y} = \frac{19}{96}$. Ans: $\boxed{(19,96)}$.

2nd Solution: By Menelaus' Theorem: $\frac{BD}{DA} \cdot \frac{AF}{FC} \cdot \frac{CE}{BE} = 1$ which yields $\frac{CE}{BE} = \frac{19}{96}$ by substitution.

Also by MT we have $\frac{AC}{CF} \cdot \frac{FE}{ED} \cdot \frac{BD}{BA} = 1$, yielding $FE = ED$.

Thus, $$\frac{a(\Delta FCE)}{a(\Delta DBE)} = \frac{\left(\frac{1}{2}\right)(CE)(EF)\sin\angle CEF}{\left(\frac{1}{2}\right)(BE)(ED)\sin\angle DEB} = \frac{CE}{BE} = \frac{19}{96}.$$

T–10. First, note that for positive integer n and real number r, $[n + r] = n + [r]$. Applied to $\left[\left[\left[x\right] + 6x\right] + 15x\right]$ we obtain $\left[\left[x\right] + \left[6x\right] + 15x\right] = \left[x\right] + \left[6x\right] + \left[15x\right]$. Applied to $N(x)$ we obtain $N(x) = \left[x\right] + \left[6x\right] + \left[15x\right] + \left[65x\right] + \left[143x\right]$. For positive integers a and b, expressions $\left[ax\right]$ and $\left[bx\right]$ increase in value by 1 at intervals of $\frac{1}{a}$ and $\frac{1}{b}$ respectively, taking on integer values from 0 to a and 0 to b on $0 \le x \le 1$. We would expect $N(x)$ to take on the 231 values from 0 to $1 + 6 + 15 + 65 + 143 = 230$, but whenever $x = \frac{p}{q}$ with p and q relatively prime and q dividing both a and b, $\left[bx\right]$ will cause an additional increase of 1, thereby causing N to skip an integer. For example, 13 divides both 65 and 143. If $x = \frac{3}{13}$, both $\left[65x\right]$ and $\left[143x\right]$ increase by 1, causing $N(x)$ to increase by 2. We say that $\left[143x\right]$ causes an additional increase or skip at $x = \frac{3}{13}$. So, $\left[6x\right]$ causes a skip at $x = 1$, $\left[15x\right]$ causes a skip at $x = \frac{1}{3}, \frac{2}{3}$ and 1, $\left[65x\right]$ causes a skip at $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, and 1, and $\left[143x\right]$ causes a skip at $x = \frac{1}{13}, \ldots, \frac{12}{13}$, and 1. There are 22 skips so N takes on $231 - 22 = \boxed{209}$ values.

ARML Power Question – 1996

Let the number 1 be placed at each end of a stick, forming list L_1 as shown:

L_1: 1 ________________________________ 1

List L_{n+1} is formed from list L_n for $n \geq 1$ by inserting between every pair of consecutive terms a_i and a_{i+1} in L_n the sum $a_i + a_{i+1}$. Lists L_2, L_3, and L_4 are shown:

L_2: 1 ________ 2 ________ 1

L_3: 1 ____ 3 ____ 2 ____ 3 ____ 1

L_4: 1 __ 4 __ 3 __ 5 __ 2 __ 5 __ 3 __ 4 __ 1

---

1. a) Prove that every positive integer appears in some L_n.

   b) Let $H(n)$ be the number of times 13 appears in list L_n. Compute the values of H for $n = 2, 6, 10, 12$, and 100. Justify your answers.

2. a) Compute the first seven members of L_{1996}.

   b) If 1995, 8, and 1997 form three consecutive terms in list L_n, compute the value of n. Justify.

3. Determine with proof a formula in terms of n for the number of terms in L_n.

4. Determine with proof a formula in terms of n for the average of the terms in L_n.

5. Number the elements in the lists from left to right. The 2 in L_3 is in the 8th position and the rightmost 3 in L_4 is in the 17th position. Compute the value of the element in the 524,320th position. Show your reasoning.

6. a) Let $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. Show that F_{n+1} is the largest element in L_n.

   b) Determine with explanation the least value of n for which 1996 could possibly appear in L_n.
   You need not prove that 1996 actually appears in that list.

7. Show that two consecutive elements of any list L_n are relatively prime.

8. Show that numbers a and b can be consecutive elements in at most one list L_n.

9. Let $N(t)$ be the number of times the number t appears in a list. Compute the maximum value of $N(1996)$. Prove your answer.

10. Prove that if n is even, then L_n has twice as many odd terms as even.

1a. L_n always begins with 1, n. This is clearly true for $n = 1$. If it's true for n, then L_{n+1} begins with $1, n+1, n$. Hence the result follows from induction.

1b. Shown are lists L_1 through L_6. Since the lists are symmetric we need only double the number of 13's that appear in the first half.

L_1	1																																1
L_2	1																2																1
L_3	1								3								2								3								1
L_4	1				4				3				5				2				5				3				4				1
L_5	1		5		4		7		3		8		5		7		2		7		5		8		3		7		4		5		1
L_6	1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	7	12	5	13	8	11	3	10	7	11	4	9	5	6	1

Clearly, $H(2) = 0$ and $H(6) = 2$. In L_7 we obtain 4 additional 13's from 9 + 4 and 10 + 3, making $H(7) = 6$. In L_8 we obtain 4 additional 13's from 7 + 6 and 2 + 11 making $H(8) = 10$. Since if two consecutive elements of a list sum to more than 13, no number generated between them will equal 13, we can write half of each successive list, omitting all numbers greater than 13 as well as all numbers sandwiched between them.

L_7	$\rightarrow$	$\{1, 7, 6, 13, 4, 13, 13, 9, 11, 2\}$	$H(7)$	=	6
L_8	$\rightarrow$	$\{1, 8, 7, 13, 6, 13, 13, 13, 11, 13, 2\}$	$H(8)$	=	10
L_9	$\rightarrow$	$\{1, 9, 13, 13, 13, 13, 13\}$	$H(9)$	=	10
L_{10}	$\rightarrow$	$\{1, 10, 13, 13, 13, 13, 13\}$	$H(10)$	=	10
L_{11}	$\rightarrow$	$\{1, 11, 13, 13, 13, 13, 13\}$	$H(11)$	=	10
L_{12}	$\rightarrow$	$\{1, 12, 13, 13, 13, 13, 13\}$	$H(12)$	=	10
L_{13}	$\rightarrow$	$\{1, 13, 13, 13, 13, 13, 13\}$	$H(13)$	=	12

We pick up no additional 13's after L_{13}, so $H(100) = 12$.

2a. $L_{1993} = \{1, 1993, \ldots\}$. Therefore, $L_{1994} = \{1, 1994, 1993, \ldots\}$, making

$L_{1995} = \{1, 1995, 1994, 3987, 1993, \ldots\}$, so $L_{1996} = \{1, 1996, 1995, 3989, 1994, 5981, 3987, \ldots\}$.

Alternately, formulas can be developed for the elements in each position:

List	3rd Element	4th	5th	6th	7th
L_4	3	5	2	5	3
L_5	4	7	3	8	5
L_6	5	9	4	11	7
L_7	6	11	5	14	9
L_n	$n-1$	$2n-3$	$n-2$	$3n-7$	$2n-5$
Thus, L_{1996}	1995	3989	1994	5981	3987

2b. If 1995, 8, 1997 appear consecutively in L_n, then 1995 – 8, 8, and 1997 – 8 must appear in L_{n-1}. Continuing backwards, $1995 - 8 \cdot 249, 8, 1997 - 8 \cdot 249 = 3, 8, 5$ must appear in L_{n-249}. Since 3, 8, 5 appears in L_5, then $n - 249 = 5$ implies $n = 254$.

3. Let $f(n)$ be the number of terms in L_n. We show by induction that $f(n) = 2^{n-1} + 1$. Clearly, $f(1) = 2^{1-1} + 1 = 2$. Assume $f(n) = 2^{n-1} + 1$. In creating L_{n+1}, we put a number in each space between consecutive numbers, and since by assumption there are 2^{n-1} spaces in L_n, then $f(n+1) = 2^{n-1} + 1 + 2^{n-1} = 2 \cdot 2^{n-1} + 1 = 2^{(n+1)-1} + 1$.

Alternately, if we ignore the right-hand 1 in each list, the number of terms doubles each time since each element receives 1 new right-hand neighbor. L_1 has 1 term (ignoring the right-hand 1) so L_n has 2^{n-1} terms. Counting the right-hand 1 gives $2^{n-1} + 1$.

4. Let $s(n)$ denote the sum of the terms in L_n. We note that $s(1) = 2, s(2) = 4$, and $s(3) = 10$, so it seems reasonable to conjecture that $s(n) = 3^{n-1} + 1$. Clearly $s(1) = 3^{1-1} + 1$. Assume that $s(n) = 3^{n-1} + 1$. The new terms added to form L_{n+1} represent sums of two consecutive terms in L_n and each member of L_n except the first and last 1 appears in exactly two of these sums. Hence, $s(n+1) = s(n) + 2\left(s(n) - 1\right) = 3\left(s(n)\right) - 2 = 3\left(3^{n-1} + 1\right) - 2 = 3^{(n+1)-1} + 1$. Thus, the average of the terms in L_n equals $\dfrac{3^{n-1} + 1}{2^{n-1} + 1}$.

<u>Alternately</u>, the sum of the terms in L_{n+1} consists of each term in L_n exactly three times except for the last 1 which appears only once. Thus, $s(n+1) = 3\left(s(n)\right) - 2 \cdot 1$ and the induction follows.

<u>Alternately</u>, we note this pattern:

$$
\begin{aligned}
s(1) &= 2 &&= 2\\
s(2) &= 3 \cdot 2 - 2 &&= 4\\
s(3) &= 3^2 \cdot 2 - 3 \cdot 2 - 2 &&= 10\\
s(4) &= 3^3 \cdot 2 - 3^2 \cdot 2 - 3 \cdot 2 - 2 &&= 28\\
\\
s(n) &= 2\left(3^{n-1} - 3^{n-2} - \ldots - 3^1 - 3^0\right) && \\
&= 2\left(3^{n-1} - \frac{3^{n-1} - 1}{2}\right) &&= 3^{n-1} + 1
\end{aligned}
$$

and the proof can proceed by induction.

5. The sum of the number of elements in lists L_1 through $L_n = (1 + 1) + (2 + 1) + (4 + 1) + \ldots + \left(2^{n-1} + 1\right) = \left(2^n - 1\right) + n$. If $n = 19$ the sum is 524,306. Thus we seek the 14th element in L_{20}. We can generate the first 17 elements in L_{20} from the first two elements in L_{16} since the increase in numbers of elements in L_1 through L_5 from two to seventeen is reflected in the increase in elements between any two consecutive numbers in subsequent lists. Since L_{16} begins 1, 16 then L_{17} begins 1, 17, 16, L_{18} begins 1, 18, 17, 33, 16, L_{19} begins 1, 19, 18, 35, 17, 50, 33, 49, 16, and so L_{20} begins 1, 20, 19, 37, 18, 53, 35, 52, 17, 67, 50, 83, 33, 82, 49, 65, and 16. The 14th element is 82.

6a. By induction. In L_1 the numbers F_1 and F_2 appear consecutively and F_2 is the largest number in L_1. Similarly, F_2 and F_3 appear in L_2 with F_3 being the largest. Assume that in list L_n the numbers F_n and F_{n+1} appear consecutively with F_{n+1} being the largest number in L_n. In list L_{n+1} the numbers $F_n + F_{n+1} = F_{n+2}$ and F_{n+1} will appear consecutively. Since each term in L_{n+1} is the sum of a term in L_n and L_{n-1}, the largest of which, by assumption, are F_{n+1} and F_n, then F_{n+2}, the sum of the two largest terms in L_n and L_{n-1}, is the largest term in L_{n+1}.

6b. Since $F_{17} = 1597$ is the largest term in L_{16}, the least value of *n* such that 1996 first appears in L_n must be greater than or equal to 17. Note: 1996 first appears in L_{18} as the sum of 741 and 1255 as well as the sum of 765 and 1231. We obtained the result by writing a calculator program, but we're still looking for a short analytical proof that $n = 18$ is the answer.

7. The elements of L_1 are relatively prime. For the inductive step, note that one of the two consecutive elements (the lesser, in fact) must have been present in a previous list. Call the elements *a* and *b* with *b* the lesser. In general, the greatest common divisor of *a* and *b* is the greatest common divisor of *a* and $a - b$. But *b* and $a - b$ are consecutive in the previous list. If *b* and $a - b$ are relatively prime, then so are *a* and *b*.

8. If *a* and *b* are consecutive terms in L_n with $a > b$, then in L_{n-1} we have the consecutive terms $a - b$ and *b*. Similarly, if $b > a - b$, then in L_{n-2} we have the consecutive terms $a - b$, and $2b - a$. In this way we can trace backwards in a unique manner until we arrive at L_1. For an example, see problem 2b. Since the number of steps in such a process is unique, *a* and *b* cannot be consecutive elements in more than one list.

Alternately, consider all ordered pairs of consecutive numbers (a_i, b_i) or (b_i, a_i) with $b_i \le a_i$ such that the pair is consecutive in lists L_n and L_m. Let a_1 be the smallest of all such a_i. Working backwards, *b* and $a_1 - b$ are consecutive in L_{n-1} and L_{m-1}, but $a_1 - b < a_1$, which is a contradiction.

9. Consider the pair of consecutive elements (*a*, *b*) or (*b*, *a*) with $b < a$. From problem 7 we know that *a* and *b* are relatively prime. From Problem 8 we know that the pair appears at most once. We must show that if the greatest common divisor of *a* and *b* is 1, then *a* and *b* appear as consecutive elements in at least one list L_n.

Proof by induction. For $a = 1$ the statement is clearly true. Suppose the gcd of a and b is 1. This implies that the gcd of a and $a - b$ is 1. If we assume that b and $a - b$ appear as consecutive elements in some list and are relatively prime, then a and b will appear as consecutive elements in the next list and be relatively prime as well.

This result, together with problems 7 and 8, implies that a pair of numbers b and 1996 can be consecutive in at most one list and that any and all numbers relatively prime to 1996 will be paired with 1996 in some list. Thus, the number of times 1996 can appear is equal to the number of numbers relatively prime to and less than 1996. Answer:

$$1996 - \frac{1996}{2} - \frac{1996}{499} + \frac{1996}{998} = 996$$

10. In terms of parity, for n even, L_n looks like 0E0 0E0 0E0 . . . . 0E0, while for n odd, L_n looks like 00E 00E 00E . . . 00E 00. We conjecture that for n even, there are twice as many odds as evens, and for n odd, there are 2 more than twice as many odds as evens, but the proof of the former requires that we show that such patterns are, indeed, generated. Thus, we have the following proof:

Proof: Since L_2 consists of $\{1, 2, 1\}$, the proposition is true for $n = 2$. Assume that for L_{2n} there are twice as many odd terms as even; thus, there are $\frac{1}{3}\left(2^{2n-1} + 1\right)$ even terms and $\frac{2}{3}\left(2^{2n-1} + 1\right)$ odd terms. Since each even term in L_{2n} has an odd term on either side of it, the even terms will generate twice their number of odd terms in L_{2n+1}, i. e., the number of new odd terms in L_{2n+1} is $2\left(\frac{1}{3}\left(2^{2n-1} + 1\right)\right)$. Each pair of consecutive odd numbers will generate an even number. Thus, the number of consecutive odd pairs in L_{2n} is $2^{2n-1} - 2\left(\frac{1}{3}\left(2^{2n-1} + 1\right)\right)$ and that equals the number of new even terms in L_{2n+1}. This gives $\frac{1}{3}\left(2^{2n} - 1\right)$ even terms and $\frac{1}{3}\left(2^{2n+1} + 4\right)$ odd terms in L_{2n+1}, implying that the number of new odd terms in L_{2n+2} is $2\left(\frac{1}{3}\left(2^{2n} - 1\right)\right)$ and the number of new even terms is $2^{2n} - 2\left(\frac{1}{3}\left(2^{2n} - 1\right)\right)$. This gives a total of $\frac{1}{3}\left(2^{2n+1} + 1\right)$ even terms and $\frac{2}{3}\left(2^{2n+1} + 1\right)$ odd terms in L_{2n+2}.

Alternately, let A_n denote the number of odd/odd consecutive pairs and B_n denote the number of odd/even or even/odd consecutive pairs in L_n. Looking at list L_{n-1}, if we have odd/odd, then the next list L_n has an even inserted, giving two even/odds. If we have an odd/even or even/odd in L_{n-1}, then L_n has an odd inserted, making an odd/odd and an even/odd. Thus, $A_n = B_{n-1}$ and $B_n = 2A_{n-1} + B_{n-1}$, giving $B_n = B_{n-1} + 2B_{n-2}$. Recalling that the Fibonacci sequence $F_n = F_{n-1} + F_{n-2}$ has the general formula $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ expressible as $C_1\left(r^n\right) + C_2\left(s^n\right)$ where r and s are roots of $t^2 = t + 1$, we conjecture that $B_n = B_{n-1} + 2B_{n-2}$ has the general formula $C_1\left(r^n\right) + C_2\left(s^n\right)$ where r and s are roots of $t^2 = t + 2$. Thus $r = -1$ and $s = 2$ making $B_n = C_1(-1)^n + C_2(2)^n$. Since $B_1 = 0$ and $B_2 = 2$, we obtain $-C_1 + 2C_2 = 0$ and $C_1 + 4C_2 = 2$, giving $C_1 = \frac{2}{3}$ and $C_2 = \frac{1}{3}$. Thus, $B_n = \frac{1}{3}\left(2(-1)^n + 2^n\right)$ and $A_n = \frac{1}{3}\left(2(-1)^{n-1} + 2^{n-1}\right)$. The number of even terms E_n is half the number of even/odds, so $E_n = \frac{1}{2}B_n = \frac{1}{3}\left((-1)^n + 2^{n-1}\right)$ and the number of odd terms O_n is $\left(2^{n-1} + 1\right) - \frac{1}{3}\left((-1)^n + 2^{n-1}\right) = \frac{2}{3}\left(2^{n-1}\right) + 1 - \frac{1}{3}(-1)^n$. In particular, for even n, $O_n = \frac{2}{3}\left(2^{n-1}\right) + \frac{2}{3} = 2 \cdot E_n$ and for odd n, $O_n = \frac{2}{3}\left(2^{n-1}\right) + \frac{4}{3} = 2 \cdot E_n + 2$.

Alternately, since in L_n there are 2^{n-1} spaces and every consecutive pair spans a space, the number of odd/odd consecutive pairs plus the number of odd/even or even/odd consecutive pairs equals 2^{n-1}. Thus $A_n + B_n = 2^{n-1}$. Since $A_n = B_{n-1}$, then $B_{n-1} + B_n = 2^{n-1}$. We obtain:

$$\begin{aligned} B_n &= 2^{n-1} - \left(B_{n-1}\right) \\ &= 2^{n-1} - \left(2^{n-2} - B_{n-2}\right) \\ &= 2^{n-1} - 2^{n-2} + \left(2^{n-3} - B_{n-3}\right) \\ &= 2^{n-1} - 2^{n-2} + 2^{n-3} - \ldots \pm \left(2 - B_1\right) \end{aligned}$$

Since $B_1 = 0$, we have a geometric progression with $n-2$ terms, a first term a equal to 2^{n-1} and a common ratio $r = -\frac{1}{2}$. Thus, $B_n = \dfrac{\left(2^{n-1}\right)\left(1-\left(-\frac{1}{2}\right)^{n-1}\right)}{1-\frac{-1}{2}} = \frac{1}{3}\left(2(-1)^n + 2^n\right)$, and we can proceed as before.

I–1. The sum of 19 consecutive positive integers equals p^3, where p is a prime number. Compute the smallest of the 19 integers.

I–2. The roots of $ax^2 + bx + c = 0$ are irrational, but their calculator approximations are 0.8430703308 and – 0.5930703308. If a, b, and c are integers whose greatest common divisor is 1 and which satisfy $a > 0$, $|b| \le 10$ and $|c| \le 10$, compute the ordered triple (a, b, c).

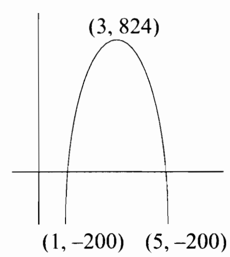
I–3. The years from 1900 to 1999 are written down consecutively and then pluses and minuses are placed alternately between the digits as shown:

$$1 + 9 - 0 + 0 - 1 + 9 - 0 + 1 - 1 + 9 - 0 + 2 - \ldots\ldots - 1 + 9 - 9 + 9 = K$$

Compute the value of K.

I–4. Given the four-digit base 10 number $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ with $A \neq 0$, let the palindromic distance L be defined by $L = |A - D| + |B - C|$. Compute the number of positive four-digit numbers where $L = 1$.

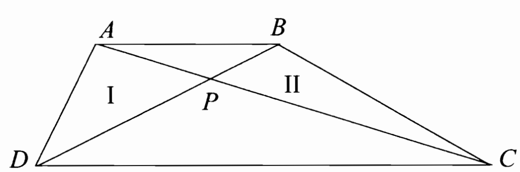
I–5. Shown is the complete graph of $y = f(x)$, a polynomial function of degree 10 whose domain is restricted to $\{x\colon 1 \le x \le 5\}$. Function f is symmetric about $x = 3$. Compute the number of solutions to the equation $f(x) = f(f(x))$.



I–6. The sides of a regular pentagon are extended to form a five-pointed star. If the ratio of the area of the pentagon to the area of the star equals $\sin\theta$, for $0° < \theta < 90°$, compute the value of θ.

I–7. Consider the following sequence of 250 numerals in base b: $1_b, 2_b, 3_b, 4_b, \ldots, 249_b, 250_b$. If $b = 25$, compute in base 10, the largest difference between the values of consecutive numerals.

I–8. In trapezoid $ABCD$, with $AB < DC$, the sum of the areas of regions I and II is 1996. If the lengths of bases $\overline{AB}$ and $\overline{CD}$ are integers and the distance between them is an integer, compute the minimum area of $ABCD$.



1. 352
2. (4, –1, –2)
3. 802
4. 332
5. 4
6. 18°
7. 391
8. 4491

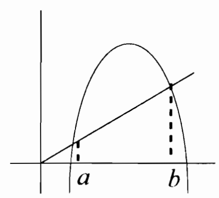
I–1. Let n be the middle term, giving $(n-9)+(n-8)+\ .\ .\ +n+\ .\ .+(n+8)+(n+9) = p^3$. Thus $19n = p^3$ making $p = 19$ and $n = 19^2$. The least term is $361-9 = \boxed{352}$.

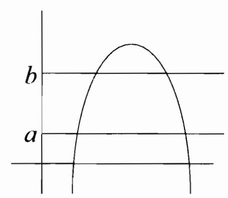
I–2. Multiplying .84307 by –.59307 gives –.4999995. . . and the assumption that b and c are small implies a product of $-\frac{1}{2}$. Adding the two roots gives $\frac{1}{4}$. The quadratic is $x^2 - \frac{x}{4} - \frac{1}{2} = 0 \rightarrow 4x^2 - x - 2 = 0$. Thus $(a, b, c) = \boxed{(4,-1,-2)}$.

I–3. There are 100 dates. Adding the 1's in the thousand's place gives $1 + 99(-1) = -98$. There are one hundred 9's in the hundred's place for a total of 900. The sum of the ten's place numbers will be $-(10 \cdot 0 + 10 \cdot 1 + \ldots + 10 \cdot 9) = -10(0+1+\ .\ .\ +9)$, but it will be cancelled by the sum of the unit's place numbers: $10(0 + 1 + \ .\ .\ + 9)$. Thus the sum is $900 - 98 = \boxed{802}$.

I–4. The conditions $|A - D| = 1$ and $B - C = 0$ give 9 ordered pairs for (A, D) from (1, 0) to (9, 8) and 8 from (1, 2) to (8, 9) for a total of 17. There are ten possibilities for (B, C) from (0, 0) to (9, 9), making a total of $17 \cdot 10 = 170$ possible numbers. The conditions $A - D = 0$ and $|B - C| = 1$ give 9 possibilities for (A, D) from (1, 1) to (9, 9), 9 possibilities for (B, C) from (1, 0) to (9, 8) and 9 possibilities for (B, C) from (0, 1) to (8, 9), for a total of $9 \cdot 18 = 162$ possible numbers. The grand total: $\boxed{332}$.

I–5. Let a and b be the two solutions to $f(x) = x$. Clearly there are two solutions to both $f(x) = a$ and $f(x) = b$. Ans: $\boxed{4}$.





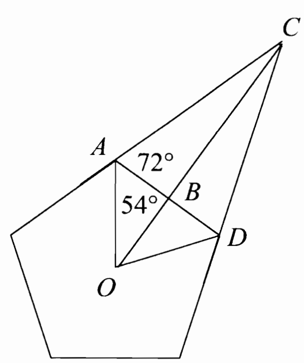
I–6. Without loss of generality let $AD = 2$, making $AB = 1$. The ratio of the area of the pentagon to the area of the star equals

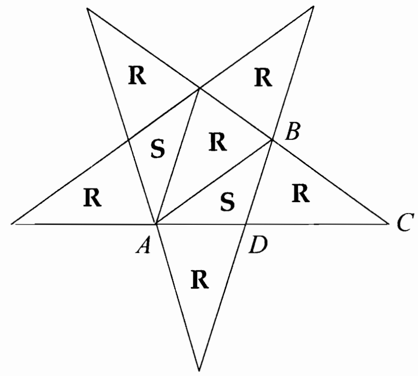
$$\frac{a(\Delta AOB)}{a(\Delta AOC)} = \frac{OB}{OC} = \frac{OB}{OB + BC} = \frac{\tan 54°}{\tan 54° + \tan 72°} \text{ since}$$

$\tan 54° = \frac{OB}{1}$ and $\tan 72° = \frac{BC}{1}$. Converting, we obtain

$$\frac{\sin 54° \cos 72°}{\sin 54° \cos 72° + \cos 54° \sin 72°} = \frac{\sin 54° \cos 72°}{\sin(54° + 72°)} =$$

$$\frac{\sin 54° \cos 72°}{\sin 126°} = \frac{\sin 54° \sin 18°}{\sin 54°} = \sin 18°. \text{ So, } \theta = \boxed{18°}.$$

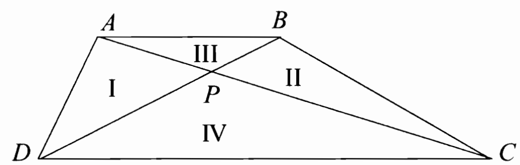




2nd Solution: The star consists of six congruent 36-72-72 triangles with area R and two congruent 36-36-108 triangles with area S. The desired ratio $= \dfrac{R + 2S}{6R + 2S}$. Let $BC = DC = 1$, then $BD = AD = \dfrac{\sqrt{5} - 1}{2}$ and $S : R = AD : DC = \dfrac{\sqrt{5} - 1}{2} : 1$ giving $S = \left(\dfrac{\sqrt{5} - 1}{2}\right) \cdot R$. Substituting into $\dfrac{R + 2S}{6R + 2S}$, we obtain $\dfrac{\sqrt{5} - 1}{4}$ which equals $\sin 18°$.

I–7. There are three possibilities as indicated by these examples. First, $188_b - 187_b = \left(b^2 + 8b + 8\right) - \left(b^2 + 8b + 7\right) = 1$. Second, $190_b - 189_b = \left(b^2 + 9b + 0\right) - \left(b^2 + 8b + 9\right) = b - 9$. Third, $200_b - 199_b = 2b^2 - \left(b^2 + 9b + 9\right) = b^2 - 9b - 9$. If $b = 25$, the difference in the values of consecutive numerals can be 1, $25 - 9 = 16$, or $25^2 - 9 \cdot 25 - 9 = \boxed{391}$.

I–8. Let m and n be the lengths of the two bases AB and DC with $m < n$, and let h be the height of the trapezoid. The distance from P to $AB = km$ and the distance from P to $DC = kn$ since

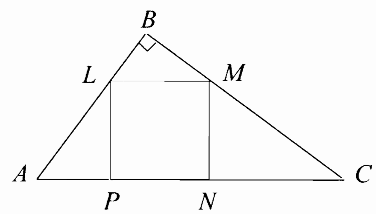


$\Delta APB \sim \Delta CPD$. From $km + kn = h$, we have $k = \dfrac{h}{m + n}$. Thus area (III) equals $\dfrac{1}{2}(km)m = \dfrac{hm^2}{2(m + n)}$ and area (IV) $= \dfrac{1}{2}(kn)n = \dfrac{hn^2}{2(m + n)}$. The area of $ABCD = \dfrac{1}{2}h(m + n) = 1996 + \dfrac{hm^2}{2(m + n)} + \dfrac{hn^2}{2(m + n)}$. Solving for h, we obtain $h = \dfrac{1996(m + n)}{mn}$. The area of $ABCD$ equals $\dfrac{998(m + n)^2}{mn} = 998\left(\dfrac{m}{n} + 2 + \dfrac{n}{m}\right)$. The minimum of $\dfrac{m}{n} + \dfrac{n}{m} = 2$ when $m = n$, but here $m < n$. To minimize $998\left(\dfrac{m}{n} + \dfrac{n}{m}\right)$, assume that m and n are relatively prime, thus $m + n$ and $m \cdot n$ are relatively prime implying that both m and n must be divisors of 1996, namely 1, 2, 4, 499, 998, or 1996. Choosing $m = 1$ and $n = 2$, $\dfrac{m}{n} + \dfrac{n}{m} = \dfrac{5}{2}$, making $h = 2994$ and the area of $ABCD = \boxed{4491}$.

ARML Relay #1 – 1996

R1–1. In trapezoid $ABCD$, $\overline{AB}$ is parallel to $\overline{CD}$ and angles B and D are acute. If $\sin\angle B = \cos\angle D$, compute $m\angle A - m\angle B$ in degrees.

R1–2. Let T = TNYWR and set $K = \frac{T}{15}$. Square $LMNP$ is inscribed in right triangle ABC as shown. If $PN = K$, compute the product $(AP)(NC)$.

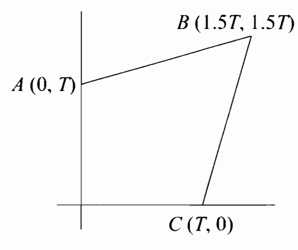


R1–3. Let T = TNYWR and set $K = \frac{T}{6}$. There are K positive integers in an arithmetic progression with common difference 1. If the sum of the terms is 75, compute the value of the first term.

ARML Relay #2 – 1996

R2–1. Let n be a positive integer. If the number of integers in the domain of $y = \log\big((1-x)(x-n)\big)$ equals $2n - 6$, compute n.

R2–2. Let T = TNYWR. Shown is the graph of the first quadrant portion of a relation. The relation is symmetric over the x-axis and over the origin. Given $A(0, T)$, $B\left(\frac{3T}{2}, \frac{3T}{2}\right)$, and $C(T, 0)$, compute the area bounded by the graph.



R2–3. Let T = TNYWR. Compute the largest integer n with $0 < n \le T$ such that the triangle with sides n, $2n - 3$, and $3n - 9$ has integer area.

Relay #1:

R1–1. 90°

R1–2. 36

R1–3. 10

Relay #2:

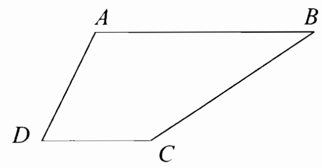
R2–1. 4

R2–2. 96

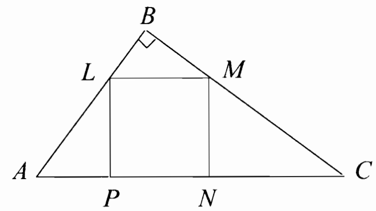
R2–3. 74

Solutions to ARML Relay #1 – 1996

R1–1. Since $\angle B$ and $\angle D$ are acute and $\sin\angle B = \cos\angle D$, then $m\angle B + m\angle D = 90°$. But $m\angle A = 180° - m\angle D$, so $m\angle A - m\angle B = (180° - m\angle D) - (90° - m\angle D) = \boxed{90°}$.



R1–2. We have $T = 90$ and $K = 6$. Since $\Delta APL \sim \Delta MNC$, $\frac{AP}{PL} = \frac{MN}{NC}$ giving $(AP)(NC) = (PL)(MN) = (PN)^2 = K^2 = \boxed{36}$. Note that $AP \cdot NC$ is invariant.

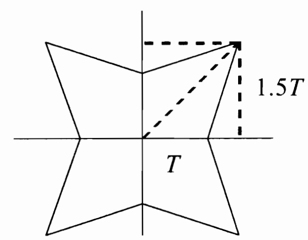


R1–3. We have $T = 36$ and $K = 6$. Letting a be the first term we have $\frac{(2a + K - 1)K}{2} = 75$. Thus, K and $(2a + K - 1)$ are factors of 150. Possible factor pairs (K, a) are (2, 37), (3, 24), (5, 13), (6, 10), and (10, 3). Since $K = 6$, then $a = \boxed{10}$.

Solutions to ARML Relay #2 – 1996

R2–1. The integers in the domain of the log function are $2, 3, \ldots n - 1$, making a total of $n - 2$ integers. Thus, $n - 2 = 2n - 6$ yields $n = \boxed{4}$.

R2–2. We have $T = 4$. The area of the first quadrant region is $2\left(\frac{1}{2} \cdot T\right)\left(\frac{3T}{2}\right) = \frac{3}{2}T^2$. Symmetry over the x-axis and origin implies symmetry over the y-axis as shown in the graph at right. The total area $= 4\left(\frac{3}{2}T^2\right) = 6T^2$. Since $T = 4$, total area $= \boxed{96}$.



R2–3. We have $T = 96$, so $0 < n \le 96$. The semiperimeter is $3n - 6$. By Heron's formula the triangle's area equals $\sqrt{(3n - 6)(2n - 6)(n - 3)3} = 3(n - 3)\sqrt{2(n - 2)}$. Clearly, the area is integral whenever $(n - 2) = 2m^2$. Since $n = 2m^2 + 2 \le 96$, then $m \le 6$ making $n = \boxed{74}$.

Note: Pass answers from position 1 to position 15.

1. If the area of a triangle with base $2N$ and height $N - 2$ is equal to N, compute N.

2. Let T = TNYWR. Compute the slope of the line passing through $(T, 1)$ and $(1, T^2)$.

3. Let T = TNYWR. If $y = Tx^2 + T^2x + C$ and the x-coordinate of the vertex equals C, compute the value of C.

4. Let T = TNYWR. If $(2 + Ti)^2 = a + bi$, compute the value of $a + b$.

5. Let T = TNYWR. Pass back the digit in the units place of the product $199^T \cdot 279^{T-1}$.

6. Let T = TNYWR. If $T^3 - 3T^2 + 3T - 1 = K^3$, compute the value of K.

7. Let T = TNYWR. If T is the number of sides of a regular polygon, compute in degrees the positive difference between the sum of the interior angles and the sum of the exterior angles.

8. Let T = TNYWR. Let $K = \dfrac{T}{60}$. If K is the height of an equilateral triangle, let the area of the triangle be A. Compute $\dfrac{A\sqrt{3}}{9}$.

9. Let T = TNYWR. If $\sin T° \cos 286° - \cos T° \sin(-106°) = \sin\theta$ for $0° \le \theta \le 180°$, then if $i = \sqrt{-1}$, compute $\cos\theta + i\sin\theta$.

10. Let T = TNYWR. Compute $\left(\dfrac{\lfloor T \rfloor}{T}\right)^2$.

11. Let T = TNYWR. Compute the value of x satisfying: $2x - \dfrac{3x - T}{2} = 7 - \dfrac{T + x}{5}$.

12. Let T = TNYWR. If $\tan 9T = \cot\theta$ for $-90° < \theta < 90°$, compute θ.

13. Let T = TNYWR. A line passes through the point $(-2, 9)$ with slope T. Compute the x-intercept of the line.

14. Let T = TNYWR. If the ordered pair (x, y) is the solution to the system below, compute $(x - y)$.

$$\begin{aligned} x + y &= T + 7 \\ Tx - 4y &= T - 8 \end{aligned}$$

15. Let T = TNYWR. A square floor is covered with square tiles. If the number of tiles in the two diagonals is $2T^2 + 5$, compute the number of tiles on the floor.

1. 3
2. – 4
3. 2
4. 8
5. 9
6. 8
7. 720°
8. 16
9. *i*
10. – 1
11. 11
12. – 9°
13. – 1
14. 4
15. 361

1. $N = \frac{1}{2}(2N)(N-2) \to N^2 - 3N = 0 \to N = \boxed{3}$.

2. $m = \dfrac{T^2 - 1}{1 - T} = -(T + 1)$. Since $T = 3$, $m = \boxed{-4}$.

3. The x-coordinate of the vertex $= \dfrac{-T^2}{2T} = \dfrac{-T}{2}$. So $C = \dfrac{-(-4)}{2} = \boxed{2}$.

4. $(2 + Ti)^2 = \left(4 - T^2\right) + 4Ti$. Hence $a + b = 4 + 4T - T^2$ which for $T = 2$ equals $\boxed{8}$.

5. Modulo 10, $199^T \cdot 299^{T-1} = 9^{2T-1} = 9^{2T+1} = 81^T \cdot 9 = 9$. Hence the last digit is $\boxed{9}$ and the value of T is irrelevant.

6. $(T-1)^3 = K^3$, so $K = T - 1 = 9 - 1 = \boxed{8}$.

7. The sum of the exterior angles is 360°, the sum of the interior angles is $(T-2)180°$, so $(T-2)180° - 360° = 180°(T-4)$. For $T = 8$, the difference equals $\boxed{720°}$.

8. $K = \dfrac{720}{60} = 12$. $A = \dfrac{K^2}{\sqrt{3}}$ so $\dfrac{A\sqrt{3}}{9} = \dfrac{K^2}{9} = \dfrac{144}{9} = \boxed{16}$.

9. $\sin T \cos 286° - \cos T \sin(-106°) = -\sin T \cos 106° + \cos T \sin 106° = \sin(106° - T)$. Since $T = 16$, $\theta = 90°$. We have $\cos 90° + i\sin 90° = \boxed{i}$.

10. $\left(\frac{\lfloor i \rfloor}{i}\right)^2 = \frac{1}{i^2} = \boxed{-1}$. Note: For real numbers other than 0 the answer is 1, for pure imaginary numbers the answer is – 1. Unfortunately, for the clever ARML participant, both answers lead to nice results down the line, and so a guess of 1 would not have led to the kind of rotten answers down the line that wrong answers usually result in.

11. $10(2x) - 5(3x - T) = 70 - 2(T + x) \rightarrow 5x + 5T = 70 - 2T - 2x \rightarrow 7x = 70 - 7T \rightarrow x = 10 - T$.
Since $T = -1$, $x = \boxed{11}$.

12. $\frac{\sin 9T}{\cos 9T} = \frac{\cos\theta}{\sin\theta} \rightarrow \sin\theta \sin 9T = \cos\theta \cos 9T \rightarrow \cos\theta\cos 9T - \sin\theta\sin 9T = 0 \rightarrow \cos(\theta + 9T) = 0$
$\rightarrow \theta + 9T = 90° + 180K \rightarrow \theta = 9(10° - T) + 180K \rightarrow \theta = 9(10 - T)$. Since $T = 11$, $\theta = \boxed{-9°}$.

13. If $y - 9 = T(x + 2)$, then for $y = 0$, $x = -\frac{9}{T} - 2$. Since $T = -9$, $x = 1 - 2 = \boxed{-1}$.

14. Multiplying the top equation by T and subtracting the bottom equation yields
$(T + 4)y = T^2 + 6T + 8 = (T + 4)(T + 2)$. Thus $y = T + 2$. So, $x + T + 2 = T + 7$, giving $x = 5$.
Then $x - y = 3 - T$. Since $T = -1$, $x - y = \boxed{4}$.

15. Let n be the number of rows in the square. If n is even, the number of tiles in the diagonals is $2n$; if n is odd, the number is $2n - 1$. Since $2T^2 + 5$ is odd, we have $2n - 1 = 2T^2 + 5 \rightarrow$
$n = T^2 + 3$. Since $T = 4$, $n = 19$, and the number of tiles is $\boxed{361}$.

1. For integers x and y with $1 < x, y \leq 100$, compute the number of ordered pairs (x, y) such that $\log_x y + \log_y x^2 = 3$.

2. Compute the number of distinct paths not passing through point (2, 2, 2) which travel from point (0, 0, 0) to point (4, 4, 4) in 12 steps, changing a coordinate by 1 at each step.

3. Compute the value of:

$$\frac{1}{2}+\frac{1}{6}\left(1^2+2^2\right)+\frac{1}{12}\left(1^2+2^2+3^2\right)+\frac{1}{20}\left(1^2+2^2+3^2+4^2\right)+\ldots+\frac{1}{3660}\left(1^2+2^2+\ldots+60^2\right)$$

1. $\log_x y + 2\log_y x = 3 \rightarrow \log_x y + \dfrac{2}{\log_x y} = 3 \rightarrow (\log_x y)^2 - 3\log_x y + 2 = 0 \rightarrow$

$(\log_x y - 1)(\log_x y - 2) = 0$. Thus, $\log_x y = 1$ or $\log_x y = 2 \rightarrow y = x$ or $y = x^2$. The solutions to the first equation are the 99 ordered pairs from (2, 2) to (100, 100); the solutions to the second are the 9 ordered pairs (2, 4), (3, 9), (4, 16), . . . , (10, 100). Thus, there are 99 + 9 = $\boxed{108}$ ordered pairs of solutions.

2. $\dfrac{12!}{4!\cdot 4!\cdot 4!} - \left(\dfrac{6!}{2!\cdot 2!\cdot 2!}\right)\cdot\left(\dfrac{6!}{2!\cdot 2!\cdot 2!}\right) = \boxed{26550}$.

3. Since $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$, then each term of the sum can be written as

$\dfrac{1}{n(n+1)} \cdot \dfrac{n(n+1)(2n+1)}{6} = \dfrac{2n+1}{6}$ for $n = 1$ to $n = 60$. Thus, the sum can be written as

$\dfrac{1}{6}(3 + 5 + 7 + \ldots + 121) = \dfrac{1}{6}\left(\dfrac{(3+121)\cdot 60}{2}\right) = \boxed{620}$.

ARML

1997

THE 22nd ANNUAL MEET

ARML celebrated its twenty-second year of competition with new teams from AAST, i.e., the Academy for the Advancement of Science and Technology, and Southwestern Pennsylvania. Including eight alternate teams, a total of eighty-nine teams and over 1335 students took part in this year's competition. This year's contest was a very difficult one and included problem #2 on the Individual Round, a question that surprisingly only 8 students nationwide were able to solve, and the equally infamous problem #8 on the Individual Round, a question that only 6 students solved correctly, making it perhaps the most difficult ARML problem ever. It was reported that #8 spawned an hour-long lecture at MSOP that summer.

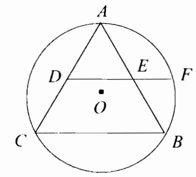
André Samson of Thomas Jefferson High School in Virginia received the Samuel Greitzer Distinguished Coach Award. André was coach of the Fairfax County ARML team from 1976 to 1992 and then coach of the TJ ARML team starting in 1993. His teams have finished in the top 4 in each of the last ten years. TJ is the only school ever to win ARML.

Sam Baethge of Texas received the Alfred Kalfus Founder's Award in recognition of his years of service to ARML in Texas. He has been the leader of the Texas teams since 1983, the sole organizer much of the time, the Southwest regional director of ARML, and a problem poser for AHSME and AIME.

David Cooper of Suffolk County and Richard Haynes of Maine received the Zachary Sobol Award in honor of their outstanding contribution to their ARML teams.

T–1. A pan of length 25 cm, width 20 cm, and height 15 cm is filled with water to a depth of 3 cm. Lead cubes of edge 4 cm are placed flat on the bottom. When the nth cube is placed in the pan, all cubes are completely covered by water for the first time. Compute n.

T–2. Equilateral triangle ABC is inscribed in circle O. Let D and E be midpoints of $\overline{AC}$ and $\overline{AB}$. If $\frac{DE}{EF}$ can be written as $\frac{a+\sqrt{b}}{c}$, for integers a, b, and c in simplest form, compute the ordered triple (a, b, c).



T–3. Let P be the parabola with vertex at the origin and directrix $y = -1$. Compute the number of lattice points on P whose distance from (0, 1) is less than or equal to 197.

T–4. In an ARML relay, the first person in group A passed back the correct answer while the first person in group B passed back an answer that was 5 more than it should have been. Problem #2 read:

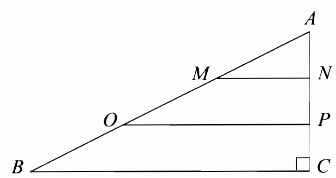
"Let K = TNYWR. If the sum of the roots of $Kx^2 + 4x + c = 0$ is S, compute $K + S$."

Both second persons in groups A and B correctly found an answer using the value of K each had received, but surprisingly, they both found the same value for $K + S$. Compute all values of $K + S$.

T–5. Note that 10 is 9 more than the sum of the squares of its digits. Compute the sum of <u>all other</u> positive two-digit base 10 integers which are 9 more than the sum of the squares of their digits.

T–6. In ΔABC, D lies on $\overline{AC}$ so that $m\angle ABD = m\angle DBC = \theta$. If $AB = 4$, $BC = 16$, and $BD = \frac{2}{\cos\theta}$, then $BD = \frac{a}{\sqrt{b}}$ for relatively prime integers a and b. Compute the ordered pair (a, b).

T–7. In ΔABC, $m\angle C = 90°$, $AC = 3\sqrt{3}$ and $BC = 6\sqrt{3}$. Line segments $\overline{MN}$ and $\overline{OP}$ are parallel to $\overline{BC}$. If the areas of ΔANM, $MNPO$, and $OPCB$ form an increasing arithmetic progression, then the length of $\overline{NP}$ has a greatest lower bound K. Compute the value of K.



T–8. Let distinct x and y be drawn from $\{1, 2, 3, \ldots, 99\}$ such that each ordered pair (x, y) is equally likely. Determine the probability that the sum of the units digits of x and y is less than 10. If the answer is $\frac{a}{b}$ for relatively prime a and b, compute the sum $a + b$.

T–9. Let $P(x)$ be a polynomial whose degree is 1996. If $P(n) = \frac{1}{n}$ for $n = 1, 2, 3, \ldots, 1997$, compute the value of $P(1998)$.

T–10. Let $ABCD$ be a regular pyramid whose edges have length 5. Consider the set of pyramids of the form $AMNP$ where M lies on $\overline{AB}$, N lies on $\overline{AC}$, and P lies on $\overline{AD}$ such that AM, AN, and $AP \in \{1, 2, 3, 4\}$. Compute the sum of the volumes of all such pyramids.

ANSWERS ARML TEAM ROUND – 1997

1. 8

2. (1, 5, 2)

3. 29

4. ± 3

5. 300

6. (8, 5)

7. $3\sqrt{2} - 3$

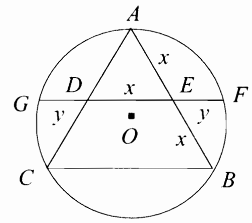
8. 7477

9. $\dfrac{1}{999}$

10. $\dfrac{250\sqrt{2}}{3}$

T–1. $3 \cdot 20 \cdot 25 + 64n > 4 \cdot 20 \cdot 25 \to 64n > 500 \to n > 7.8125 \to n = \boxed{8}$.

T–2. Extend $\overline{DE}$ until it meets the circle at G. Let $AB = 2x$ making $AE = EB = DE = x$. Let $GD = EF = y$. Thus, $DE : EF = x : y$. By the Power of a Point Theorem,



$AE \cdot EB = GE \cdot EF \to x^2 = (y + x)y \to x^2 - xy - y^2 = 0 \to$

$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0$. Solving for $\frac{x}{y}$, we obtain $\frac{1+\sqrt{5}}{2}$, so $(a, b, c) = \boxed{(1, 5, 2)}$.

T–3. The equation of the parabola is $y = \frac{x^2}{4}$ and x must be even, so let $x = 2k$. The distance from $Q\left(2k, k^2\right)$ to

$(0, 1) = \sqrt{(2k - 0)^2 + \left(k^2 - 1\right)^2} = \sqrt{k^4 + 2k^2 + 1} = k^2 + 1 \le 197$. Thus, $k^2 \le 196$ making

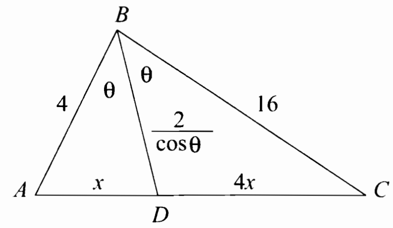
$k \in \{-14, -13, \ldots, 13, 14\}$, giving $\boxed{29}$ lattice points.

T–4. Since $S = -\frac{4}{K}$, then $K + S = K - \frac{4}{K}$. If K is the correct answer, then $K + 5$ is the incorrect answer and we have: $K - \frac{4}{K} = (K + 5) - \frac{4}{K+5} \to K^2 + 5K + 4 = 0$. Thus, $K = -1$, or -4. If the second person in A receives -1 or -4, then the second person in B receives 4 or 1 respectively, and S for A equals 4 or 1 while S for B equals -1 or -4 respectively, yielding $K + S = \boxed{\pm 3}$.

T–5. Let $10a + b$ represent the number. We seek solutions to $10a + b = a^2 + b^2 + 9$. Complete the square and multiply by 4 to eliminate fractions. Thus, $(2a - 10)^2 + (2b - 1)^2 = 65$. Since $8^2 + 1^2 = 7^2 + 4^2 = 65$, we pair $2a - 10$ with ± 8 and $2b - 1$ with ± 1 to obtain $a = 9$ or 1 while $b = 1$ or 0 giving 91, 90, 11, and 10. Similarly, pair $2a - 10$ with ± 4 and $2b - 1$ with ± 7 to obtain $a = 7$ or 3 while $b = 4$, giving 74 and 37. Hence, $11 + 34 + 74 + 90 + 91 = \boxed{300}$.

<u>Alternate solution:</u> From $10a + b = a^2 + b^2 + 9$ we obtain $(a - 1)(9 - a) = b(b - 1)$. To obtain the solutions check which of the following products can also be written as $b(b - 1)$: $0 \cdot 8, 1 \cdot 7, 2 \cdot 6, 3 \cdot 5$, or $4 \cdot 4$. Clearly, $0 \cdot 8$ and $2 \cdot 6$ work. Now choose all combinations of a and b.

T–6. By the Angle Bisector Theorem, if $AD = x$, then $DC = 4x$. By the Law of Cosines, $16x^2 = \dfrac{4}{\cos^2\theta} + 256 - 2\cdot 16\cdot\dfrac{2}{\cos\theta}\cdot\cos\theta$.

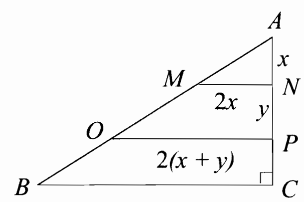


Thus, $16x^2 = \dfrac{4}{\cos^2\theta} + 192$. Likewise,

$x^2 = 16 + \dfrac{4}{\cos^2\theta} - 2\cdot 4\cdot\dfrac{2}{\cos\theta}\cdot\cos\theta = \dfrac{4}{\cos^2\theta}$. Substituting into the first equation we obtain

$16x^2 = x^2 + 192$, giving $x = \dfrac{8}{\sqrt{5}}$ → the answer is the ordered pair $\boxed{(8,5)}$.

T–7. Let $AN = x$ and $NP = y$ → $MN = 2x$ and $OP = 2(x + y)$. Then



$a(\Delta ANM) = x^2$, $a(MNPO) = (x + y)^2 - x^2$, and

$a(OPCB) = \left(3\sqrt{3}\right)^2 - \left(x + y\right)^2$. The condition that these areas form an increasing arithmetic progression yields $a(MNPO) - a(\Delta ANM) = a(OPCB) - a(MNPO)$. Thus,

$0 < (x + y)^2 - 2x^2 = 27 - 2(x + y)^2 + x^2$. Solving, we obtain $(x + y)^2 - x^2 = 9$ →

$y = \sqrt{9 + x^2} - x$ which can be rewritten as $y = \dfrac{9}{\sqrt{9 + x^2} + x}$, signifying that y is a decreasing function of x. Now we seek bounds on x. Since $(x + y)^2 > 2x^2$ and $(x + y)^2 = x^2 + 9$, then

$9 + x^2 > 2x^2 \rightarrow x < 3$. Since y is a decreasing function of x on the interval (0, 3), the greatest lower bound is at $x = 3$, i.e., at $\sqrt{18} - 3 = \boxed{3\sqrt{2} - 3}$.

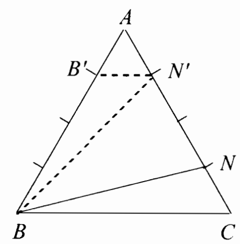
T–8. There are 9 ways for x to have 0 as a units digit and then all 98 choices for y work. For units digits n between 1 and 4, there are 10 choices for x and $98 - 10n$ choices for y. For example, if $x = 12$, we delete 12 and all 20 numbers ending in 8 or 9. For units digits n from 5 to 9, there are again 10 choices for x but $99 - 10n$ choices for y since in this case we don't delete the number itself. Adding these yields $9 \cdot 98 + 10(88 + 78 + 68 + 58) + 10(49 + 39 + 29 + 19 + 9) = 5252$. This double counts (x, y), so there are 2626 distinct pairings which work. There are a total of ${}_{99}C_2 = 4851$ pairings. The probability is $\dfrac{2626}{4851}$. The sum of 2626 and 4851 is $\boxed{7477}$.

T–9. If $P(n) = \frac{1}{n}$ for $n = 1, 2, \ldots, 1997$, then $xP(x) - 1$ has roots $x = 1, 2, \ldots, 1997$. Thus,

$xP(x) - 1 = c(x-1)(x-2)\cdot \ldots \cdot(x-1997)$. If $x = 0$, then $-1 = c(-1997!)$ so $c = \frac{1}{1997!}$.

Therefore, $1998 \cdot P(1998) - 1 = \frac{1}{1997!}(1997!) \rightarrow P(1998) = \frac{2}{1998} = \boxed{\frac{1}{999}}$.

T–10. Consider the analogous case in 2-space: ΔABN lies within equilateral triangle ABC of side 4. If N is moved to N', then the height of

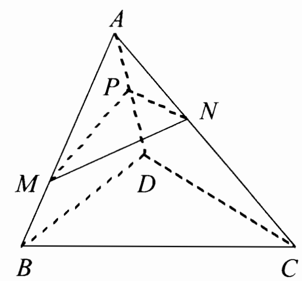


$\Delta ABN' = \frac{1}{AN}$ times the height of ΔABN. If B is moved to B', the

height of $\Delta AB'N' = \frac{1}{AM}$ times the height of $\Delta ABN'$.

$\Delta AB'N'$ is a unit triangle whose area is $\left(\frac{1}{AM}\right)\cdot\left(\frac{1}{AN}\right)$ times the area of ΔABN. Alternately, the area of $\Delta ABN = (AM)(AN)$ times the area of unit $\Delta AB'N'$.

In the three-dimensional case, let the volume of $AMNP$ be V_i. If M, N, and P are moved so that $AM = AN = AP = 1$, then $AMNP$ is reduced to a unit pyramid of edge-length 1, and its volume is reduced by $\left(\frac{1}{AM}\right)\cdot\left(\frac{1}{AN}\right)\cdot\left(\frac{1}{AP}\right)$. Let the volume of the unit pyramid be V_0.



Then $V_0 = \frac{V_i}{AM \cdot AN \cdot AP} \rightarrow V_i = (AM)(AN)(AP)V_0$.

The sum of the volumes of all the pyramids is the sum of all possible products $(AM)(AN)(AP)V_0$ for $AM, AN, AP \in \{1, 2, 3, 4\}$. This sum can be conveniently expressed as $V_0(1+2+3+4)^3 = 1000V_0$.

It remains to calculate the volume of a regular tetrahedron of side 1. Its base has area $\frac{\sqrt{3}}{4}$ and its height can be found to be $\frac{\sqrt{2}}{\sqrt{3}}$, so its volume $= \frac{1}{3}Bh = \frac{\sqrt{2}}{12}$, making $1000V_0 = \boxed{\frac{250\sqrt{2}}{3}}$.

ARML Power Question – 1997

<u>Definition #1</u>: if ΔABC is equilateral and ΔDAB, ΔDBC, and ΔDAC are all isosceles right triangles with the right angle at D, then $DABC$ is a <u>corner pyramid with apex D</u>.

<u>Definition #2</u>: a <u>cornered polygon</u> is one of the following:

a) An equilateral triangle with a corner pyramid completely covering it.

b) A regular hexagon divided into 6 equilateral triangles with corner pyramids completely covering each triangle.

c) A square to which is first attached a pyramid whose base is congruent to the square and whose faces are equilateral triangles. Each triangular face of the pyramid is then cornered. See def. 2a and problem 2.

d) A regular pentagon to which is first attached a pyramid whose base is congruent to the pentagon and whose faces are equilateral triangles. Each triangular face of the pyramid is then cornered.

<u>Definition #3</u>: a <u>cornered polyhedron</u> is a polyhedron whose faces have been replaced by the corresponding cornered polygons.

---

1. Let $DABC$ be a corner pyramid with apex D. If M is the midpoint of $\overline{AB}$, P is the foot of the altitude from D to ΔABC, and $AB = 1$, compute:

   a) DM b) MP c) DP d) $\cos \angle DMP$

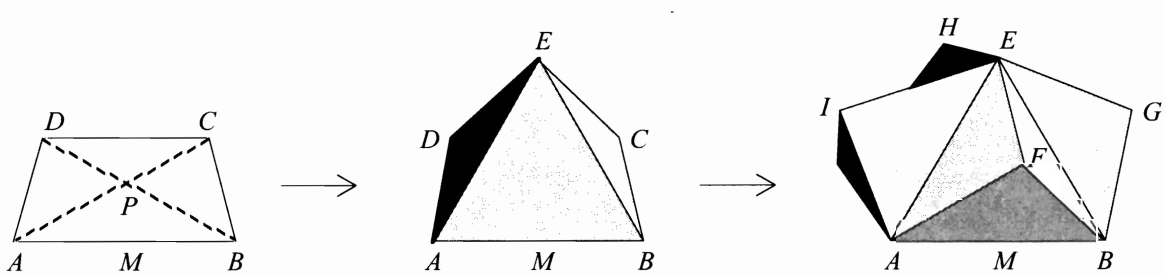
2. On the left is square $ABCD$ of side 1. P is the center of the square and M is the midpoint of $\overline{AB}$. On the right is the cornered square $ABCD$. Points F, G, H, I are the apexes of the corner pyramids. Compute:

   a) $\cos \angle FMP$.

   b) cosine of the angle formed by planes FEB and GEB.

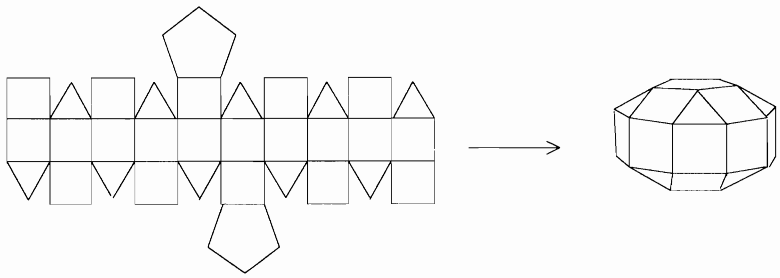
   c) perimeter of $FGHI$.

   d) volume of the cornered square $ABCD$.

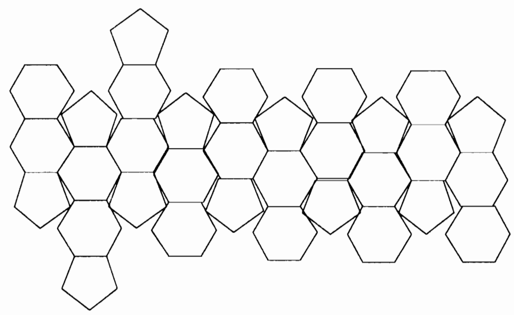


3. Let P be a right pentagonal prism whose 5 lateral faces are congruent squares. Prove that it is possible to corner P; i.e., show that the cornered faces do not overlap. The proof may use calculator approximations.

4. Shown below is a squat but instructive polyhedron. Prove that it is impossible to corner this polyhedron; that is, show that the cornered faces do overlap. The proof may use calculator approximations.



5. The pattern at the right can be folded into a soccer ball-like polyhedron called a buckyball. Compute the number of its edges and vertices.



6. Compute the number of faces, edges, and vertices of a cornered buckyball.

7. A regular tetrahedron is a pyramid all of whose faces are equilateral triangles. Prove that a cornered regular tetrahedron is a cube.

8. A regular octahedron has 8 faces which are equilateral triangles with 4 faces meeting at each of its 6 vertices. Prove that a cornered octahedron has double the volume of the original octahedron.

9. Cover each face of an octahedron with a regular tetrahedron and then corner the resulting figure. Prove that the cornered figure is a cube.

10. Let $P_1, P_2, P_3, \ldots$ be a sequence of polyhedra where P_1 is a regular tetrahedron of side 1. Create polyhedron P_{n+1} from P_n as follows: To each equilateral triangle face of P_n, attach a regular tetrahedron with vertices at the midpoints of the sides of the faces of P_n.

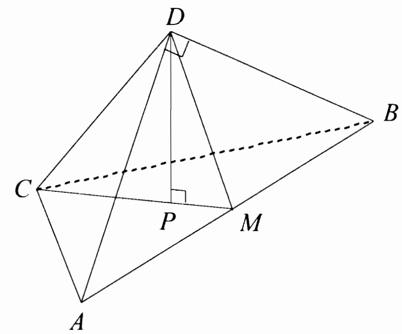
a) Compute the number of faces of P_n.

b) Compute the surface area of P_n.

c) Find the limit, as n goes to infinity, of the volume of P_n.

d) Prove that every P_n fits inside a cube of side $\dfrac{1}{\sqrt{2}}$.

1. a) $\overline{DM}$ is an altitude of isosceles right triangle DAB, so $DM = \frac{1}{2}$.



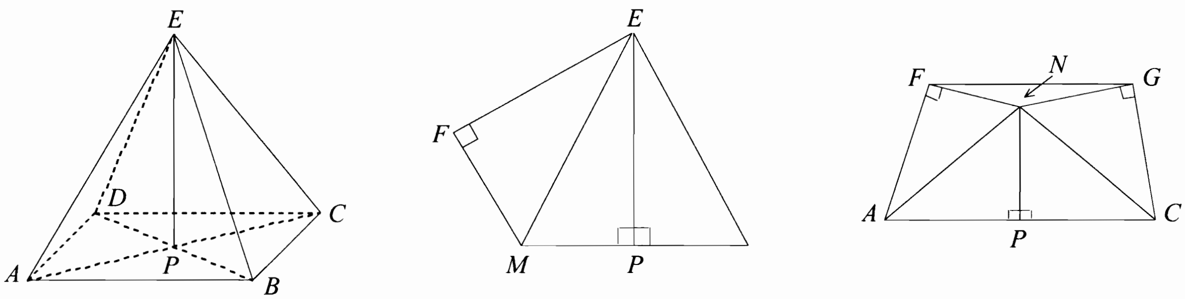
b) $MC = \frac{\sqrt{3}}{2}$ and $\frac{CP}{MP} = \frac{2}{1}$, so $MP = \frac{1}{2\sqrt{3}}$.

c) $MP^2 + DP^2 = DM^2$, so $DP = \sqrt{\frac{1}{4} - \frac{1}{12}} = \frac{1}{\sqrt{6}}$.

d) $\cos\angle DMP = \frac{MP}{DM} = \frac{1}{\sqrt{3}}$.

2. a) Pyramid $EABCD$ is the union of the four corner pyramids $PABE$, $PBCE$, $PCDE$, and $PDAE$. Define θ to be the base angle of a corner pyramid so that $m\angle FME = m\angle EMP = \theta$. By (1d), $\cos\theta = \frac{1}{\sqrt{3}}$, but then

$$\cos\angle FMP = \cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{1}{\sqrt{3}}\right)^2 - 1 = -\frac{1}{3}.$$



b) Let M be the midpoint of $\overline{EB}$. Because triangles FNA, PNA, PNC, and GNC are corresponding cross-sections of congruent corner pyramids, $m\angle FNA = m\angle PNA = m\angle PNC = m\angle GNC = \theta$, so

$$\cos\angle FNG = \cos(2\pi - 4\theta) = \cos(2(2\theta)) = 2\cos^2(2\theta) - 1 = = 2\left(-\frac{1}{3}\right)^2 - 1 = -\frac{7}{9}.$$

c) Using the diagram in 2b, $FN = GN = \frac{1}{2}$, so $FG = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - 2\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{-7}{9}\right)} = \frac{2\sqrt{3}}{3}$ by the Law of Cosines. By symmetry, the perimeter of square $FGHI$ is $\frac{8\sqrt{2}}{3}$.

Alternately, let J be the projection of point F onto the plane $ABCD$. The distance JM equals

$MF \cdot \cos \angle FMP = \frac{1}{2} \cdot \frac{-1}{3} = -\frac{1}{6}$; the negative sign means that J is outside square $ABCD$. Thus,

$FH = 2(JM + MP) = \frac{4}{3}$ and the perimeter of $FGHI$ is $\frac{8\sqrt{2}}{3}$.

d) The volume of the cornered square is 8 times the volume of a corner pyramid with base side 1 giving

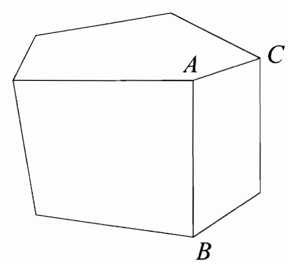
$8 \cdot \frac{1}{3} \cdot (bh) = \frac{8}{3} \cdot \frac{\sqrt{3}}{4} \cdot \frac{1}{\sqrt{6}} = \frac{\sqrt{2}}{3}$.

Alternately, the volume of the cornered square is twice the volume of pyramid $EABCD$ giving

$2 \cdot \frac{1}{3} \cdot 1 \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{3}$.

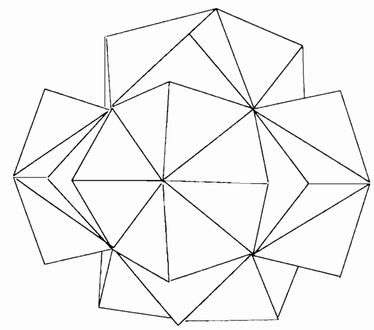
3. The question of whether one can corner a given polyhedron arose in modular origami. It is relatively easy to fold corner pyramids from 3 small squares of paper, but which polyhedra can then be built from corner pyramids? For more information, see *Multidimensional Transformations: Unit Origami* by Tomoko Fusè, Japan Publications, 1990.

The pentagonal prism at the right has two types of edges, those that join two squares such as $\overline{AB}$ and those that join a square and a pentagon such as $\overline{AC}$. First, we consider edges like $\overline{AB}$. The solid angle at $\overline{AB}$ is 108° and we hope to add two cornered squares at $\overline{AB}$.

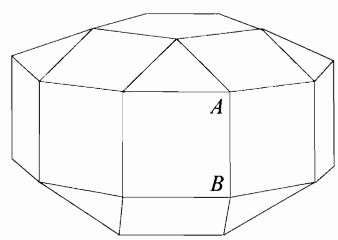


By (2a), the base angle of a cornered square is $\arccos\left(-\frac{1}{3}\right) = 90° + \arcsin\left(\frac{1}{3}\right)$. With a calculator we can approximate $\arcsin\left(\frac{1}{3}\right)$ as 19.471°. Or we could note that $\arcsin\left(\frac{1}{3}\right) < \arcsin\left(\frac{1}{2}\right) = 30°$. Either way, we can conclude that $108° + 2\left(90° + \arcsin\left(\frac{1}{3}\right)\right) < 360°$, so there is no overlap.

Alternately, note that we can fit in the belt of cornered squares only if the sum of all their base angles is less than the sum of the exterior angles of the pentagon which is $5(360° - 108°) = 1260°$. The sum of the 10 base angles of the cornered squares is $10\arccos\left(-\frac{1}{3}\right) = 900° + 10\arcsin\left(\frac{1}{3}\right)$. Because $\arcsin\left(\frac{1}{3}\right) \approx 19.471° < 30°$, it is true that $10\arccos\left(-\frac{1}{3}\right) < 1260°$.



We still need to show that there is no overlap at edges like $\overline{AC}$. The solid angle at $\overline{AC}$ is 90°. We hope to add a cornered square and a cornered pentagon at $\overline{AC}$. The pyramid attached to the pentagon during the first stage of cornering has a shorter altitude than the pyramid attached to a square during the first stage of cornering since the two altitudes have lengths $\sqrt{\frac{3}{4} - \frac{\tan^2(54°)}{4}} \approx .5257$ and $\frac{\sqrt{2}}{2} \approx .7071$ respectively. During the second stage, corner pyramids are attached in both cases. Thus, the base angle of a cornered pentagon is smaller than the base angle of a cornered square (the angles are approximately 91.113° and 109.471°, respectively). The total of the angles at $\overline{AC}$ is bounded by $90° + 2(120°) = 330°$, so there is no overlap. Thus, it is possible to corner a pentagonal prism.



4. Let P be the given polyhedron. Notice that the band of squares around the "equator" of P forms a right decagonal prism. Let $\overline{AB}$ be an edge joining 2 of these squares. The solid angle at $\overline{AB}$ is 144°. If it were possible to corner P, then the cornered squares attached to the squares adjacent to $\overline{AB}$ would be able to fit around $\overline{AB}$.

By (2a), the base angle of a cornered square is $\arccos\left(-\frac{1}{3}\right)$. With a calculator we compute $\arccos\left(-\frac{1}{3}\right)$ to be approximately 109.471° or we could note that since $\frac{1}{3} > \frac{\sqrt{5}-1}{4}$, then $\arccos\left(-\frac{1}{3}\right) > \arccos\left(\frac{1-\sqrt{5}}{4}\right) = 108°$. Either way, we can conclude that the total solid angle would be strictly greater than $144° + 2(108°) = 360°$ which is impossible. Thus, P cannot be cornered.

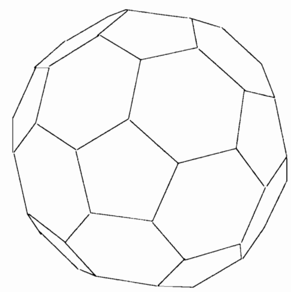
Alternately, note that all 10 cornered squares fit in only if the sum of all their base angles is less than the sum of the exterior angles of the decagon which is $12 \cdot 180° = 2160°$. The sum of the 20 base angles of the cornered squares is $20 \arccos\left(-\frac{1}{3}\right) = 20\left(90° + \arcsin\left(\frac{1}{3}\right)\right) = 1800° + 20\arcsin\left(\frac{1}{3}\right)$. Let $\alpha = \arcsin\left(\frac{1}{3}\right)$.

We know from (3) that $\alpha < 30°$. Because $\cos\alpha = \frac{2\sqrt{2}}{3}$, $\cos(2\alpha) = 2\left(\frac{8}{9}\right) - 1 = \frac{7}{9}$ and

$\cos(4\alpha) = 2\left(\frac{49}{81}\right) - 1 = \frac{17}{81}$. Because $\frac{17}{81} < \frac{1}{3} = \cos(90° - \alpha)$, we must have $\alpha + 4\alpha = 5\alpha > 90°$, so

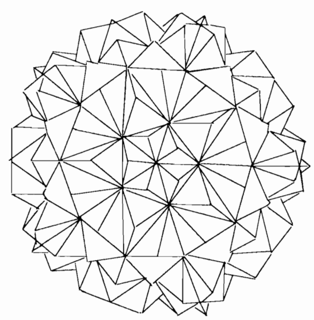
$20\alpha > 360°$. But then $20 \arccos\left(-\frac{1}{3}\right) = 20(90° + \alpha) > 2160°$. Thus, the cornered squares must overlap.

5. A buckyball has 12 pentagonal faces and 20 hexagonal faces. Three faces meet at each vertex and 2 faces meet at each edge, so a buckyball has $\frac{5 \cdot 12 + 6 \cdot 20}{2} = 90$ edges and $\frac{5 \cdot 12 + 6 \cdot 20}{3} = 60$ vertices. We can check with Euler's Formula: $V - E + F = 60 - 90 + 32 = 2$, as it should. Note that buckyballs are named after R. Buckminster Fuller,

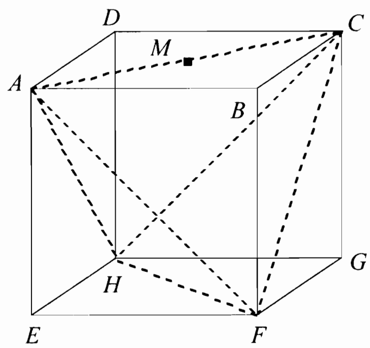


famous for his geodesic domes, and that carbon compounds with this geometry were discovered by the 1996 Nobel Prize winners in chemistry, Robert F. Curl, Harold W. Kroto, and Richard E. Smalley.

6. A cornered pentagon has 15 faces and a cornered hexagon has 18 faces, so a cornered buckyball has $15 \cdot 12 + 18 \cdot 20 = 540$ faces. A cornered pentagon has 20 additional edges and a cornered hexagon has 24 additional edges, so a cornered buckyball has $90 + 20 \cdot 12 + 24 \cdot 20 = 810$ edges. A cornered pentagon has 6 additional vertices and a cornered hexagon has 7 additional vertices, so a cornered buckyball has $60 + 6 \cdot 12 + 7 \cdot 20 = 272$ vertices. If we plug these numbers into Euler's Formula, we obtain $V - E + F = 272 - 810 + 540 = 2$ as hoped. The solid pictured, while somewhat more pointed than a cornered buckyball, has the same arrangement of vertices, edges, and faces.



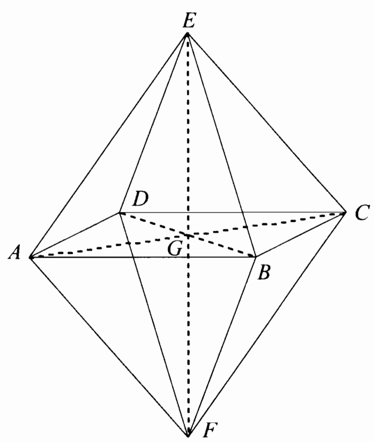
7. Let *ABCDEFGH* be a cube. Then *ACFH* is a regular tetrahedron and *BACF*, *DACH*, *EAFH*, and *GCFH* are corner pyramids, so *ABCDEFGH* is a cornered tetrahedron.



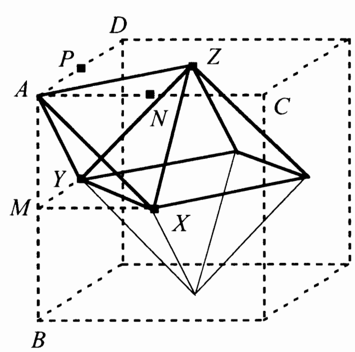
Alternately, start with tetrahedron *ACFH* and add corner pyramids *BACF*, *DACH*, *EAFH*, and *GCFH*. Let *M* be the midpoint of $\overline{AC}$. Then $\angle DMH$ and $\angle BMF$ are both base angles of corner pyramids, so $m\angle DMH = m\angle BMF = \theta$.

Thus, $m\angle DMB = 2\theta + m\angle FMH$ where $\angle FMH$, the base angle of a tetrahedron, satisfies $\cos\angle FMH = \frac{1}{3} = -\cos 2\theta$ and $\sin\angle FMH = \frac{2\sqrt{2}}{3} = \sin 2\theta$. We conclude that $m\angle FMH = 180° - 2\theta$, so $m\angle DMB = 180°$, making *ABCD* a square. Arguing similarly for the other edges of *ACFH* completes the proof.

8. Let *ABCDEF* be a regular octahedron with center *G*. Then *GABE*, *GBCE*, *GCDE*, *GDAE*, *GABF*, *GBCF*, *GCDF*, and *GDAF* are 8 congruent corner pyramids. When we corner *ABCDEF* we attach 8 more congruent corner pyramids to the faces of the octahedron, so the total volume is doubled.



9. The centers of the faces of a cube form a regular octahedron. Let *A*, *B*, *C*, and *D* be vertices of a cube, and let *X*, *Y*, and *Z* be the centers of the neighboring faces as shown. Let *M*, *N*, and *P* be the midpoints of $\overline{AB}$, $\overline{AC}$, and $\overline{AD}$ respectively. Then *AXYZ* is a regular tetrahedron (attached to face *XYZ* of the inscribed octahedron), and *MAXY*, *NAXZ*, and *PAYZ* are corner pyramids attached to the outer faces of tetrahedron *AXYZ*. Together they completely fill the *A*-octant of the cube. Reasoning similarly for each corner of the cube gives the full dissection desired.



10. a) P_1 has 4 faces. As we go from P_n to P_{n+1}, each face is replaced by 6 faces. Thus, P_n has $4 \cdot 6^{n-1}$ faces.

b) A face of P_n is an equilateral triangle of side $\frac{1}{2^{n-1}}$ which has an area of $\left(\frac{1}{4^{n-1}}\right) \cdot \left(\frac{\sqrt{3}}{4}\right) = \frac{\sqrt{3}}{4^n}$. Thus,

P_n has a total surface area of $4\left(6^{n-1}\right) \cdot \left(\frac{\sqrt{3}}{4^n}\right) = \sqrt{3} \cdot \left(\frac{3}{2}\right)^{n-1}$.

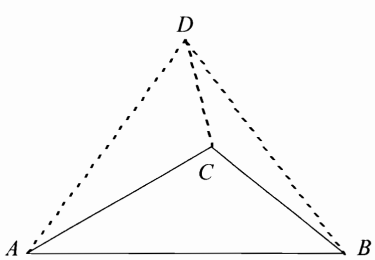
c) The volume of P_1, a tetrahedron of side 1, is $\frac{1}{3} \cdot \frac{\sqrt{3}}{4} \cdot \frac{2}{\sqrt{6}} = \frac{1}{6} \cdot \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{6\sqrt{2}}$. Going from P_n to P_{n+1},

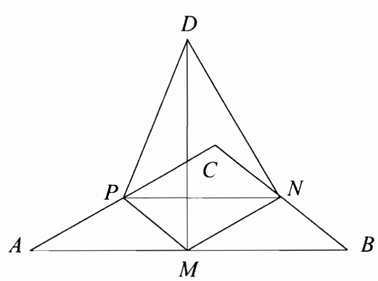
we add $4 \cdot 6^{n-1}$ tetrahedra of side length $\frac{1}{2^n}$ so $\operatorname{vol}\left(P_{n+1}\right) = \operatorname{vol}\left(P_n\right) + 4 \cdot 6^{n-1} \cdot \left(\frac{1}{8^n}\right) \cdot \frac{1}{6\sqrt{2}}$.

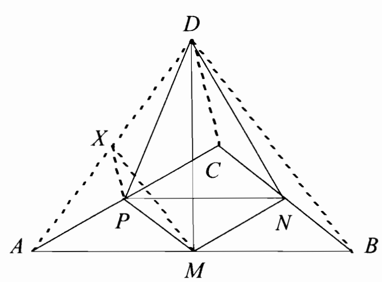
Summing from 1 to n we have $\operatorname{vol}\left(P_{n+1}\right) = \frac{1}{6\sqrt{2}} + \frac{1}{9\sqrt{2}} \cdot \sum_{k=1}^{n} \left(\frac{3}{4}\right)^k$. Thus, as $n \to \infty$, we have

$$\operatorname{vol}\left(P_{n+1}\right) \to \frac{1}{6\sqrt{2}} + \frac{1}{9\sqrt{2}} \cdot \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k = \frac{1}{6\sqrt{2}} + \frac{1}{3\sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

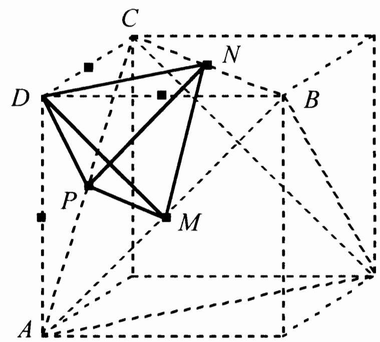
d) In fact, a cornered P_n is a cube of side $\frac{1}{\sqrt{2}}$. Questions (7) and (9) show this for P_1 and P_2, providing base cases for our induction. We complete the proof by showing that a cornered face of P_n and the cornered version of the corresponding configuration in P_{n+1} are the same; thus, the cornered versions of the P_n and P_{n+1} are the same polyhedron, namely a cube of side $\frac{1}{\sqrt{2}}$.





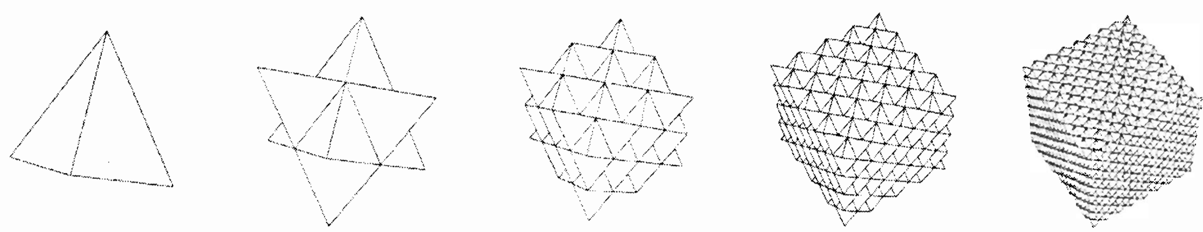


Let ABC be a face of P_n. Let M, N, and P be the midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. Let D be the apex of the tetrahedron with base MNP, so that $ABCDMNP$ is the corresponding region of P_{n+1}. Notice that D is also the apex of corner pyramid *DABC*. Let *X* be the midpoint of $\overline{AD}$. Then *X* is the apex of the corner pyramids *XAMP* and *XDMP*. Arguing symmetrically for $\overline{BD}$ and $\overline{CD}$, we see that the cornering of the configuration *ABCDMNP* is the same as the cornering of *ABC*.



Alternatively, if we let P_1 be situated inside a cube of side $\frac{1}{\sqrt{2}}$ as in (7), then P_2 will have as new vertices the rest of the vertices of the cube together with the centers of the faces of the cube. Let *ABC* be a face of P_1. Let *D* be the vertex of the cube surrounded by *A*, *B*, and *C*. Let *M*, *N*, and *P* be the midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. Then *DMNP* is one of the tetrahedra added at stage 1. It is situated within the *D*-octant of the cube in exactly the same was as P_1 is situated inside the original cube. The three pyramids that will be attached to *DMNP* at stage 2 will lie inside the *D*-octant, just as the pyramids attached at stage 1 lie inside the original cube. This is true for each octant of the original cube and so we can apply the same analysis to each of them at the next state, and so on all the way down. Since no added tetrahedron leaves its cubelet, no P_n leaves the original cube.

Shown below are P_1, P_2, P_3, P_4, and P_5. For $n \geq 3$, some of the edges of the added tetrahedra coincide. However, they never overlap along faces or have common volume. If we let $P_\infty = \bigcup_{n=1}^{\infty} P_n$, then P_∞ is *not* the entire cube. There will be uncountably many points of the cube that are outside all the P_n's. Collectively, they have volume 0.



I–1. Let a, b, c, and n be positive integers. If $a + b + c = (19)(97)$ and $a + n = b - n = \dfrac{c}{n}$, compute the value of a.

I–2. ΔABC is inscribed in circle O, the radius of O is 12 and $m\angle ABC = 30°$. A circle with center B is drawn tangent to the line containing $\overline{AC}$. Let R be the region which is within ΔABC, but outside circle B. Compute the maximum area of R.

I–3. A team wins 3 games, then loses 1, then wins 3 and loses 2, then wins 3 and loses 3, and so on, each time winning 3 games before losing one more than before. If N is the number of games played, find the least value of N such that the percentage of wins is below 25%.

I–4. Let $f(x) = 4x - x^2$. Consider the sequence $x_1, x_2, x_3, \ldots$ where $x_i = f\left(x_{i-1}\right)$ for $i > 1$. Compute the number of values of x_1 such that x_1, x_2, and x_3 are all distinct, but $x_i = x_3$ for all $i > 3$.

I–5. The sum of 1999 positive numbers in an increasing arithmetic progression is 1. Compute the width of the smallest interval containing all possible values of the common difference. Do not leave your answer in factored form.

I–6. Compute the remainder when $\cot^{1997}\left(\dfrac{\pi}{12}\right) + \tan^{1997}\left(\dfrac{\pi}{12}\right)$ is divided by 9.

I–7. The diagonals of a convex polygon connect non-adjacent vertices. The diagonals of a regular 10-sided polygon are drawn. Compute the number of parallel pairs of diagonals.

I–8. If $\sqrt[3]{\sqrt[3]{2} - 1}$ is written as $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$ where a, b, and c are rational numbers, compute the sum $a + b + c$.

1. 80

2. $\dfrac{108}{\pi}$

3. 217

4. 3

5. $\dfrac{1}{1997001}$

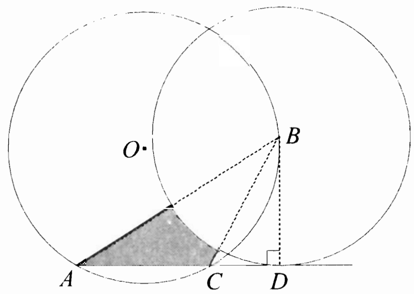
6. 4

7. 45

8. $\dfrac{1}{3}$

I–1. Since $b = a + 2n$, $c = an + n^2$, then $2a + 2n + an + n^2 = 2(a + n) + n(a + n) = (n + 2)(a + n) = 19 \cdot 97$. Thus $n + 2 = 1, 19, 97$, or 1843, but if $n + 2 = 1$ then $n < 0$ and if $n + 2 = 97$ or 1843, then $a < 0$. Thus, $n + 2 = 19 \rightarrow n = 17 \rightarrow a + 17 = 97 \rightarrow a = \boxed{80}$.

I–2. Since $m\angle ABC = 30°$, $m\angle AOC = 60°$, making $AC = 12$. The area of the shaded region $= \frac{1}{2} \cdot 12 \cdot BD - \frac{1}{12} \cdot \pi \cdot BD^2$. This is quadratic in BD and takes its maximum at $BD = \frac{-6}{-2\pi / 12} = \frac{36}{\pi}$. Maximum area $= 6\left(\frac{36}{\pi}\right) - \frac{\pi}{12} \cdot \frac{36^2}{\pi^2} = \boxed{\frac{108}{\pi}}$. Note: The maximum occurs when $\overline{BD}$ lies outside the triangle.



Figures 1, 2, and 3 below show various positions for B. When B is very close to A or C, the shaded area is relatively small. When B is on the perpendicular bisector of $\overline{AC}$, the shaded area is quite small. So clearly the shaded area increases until it reaches approximately the position in Figure 2 and then the area decreases until Figure 3. The maximum area is reached when $m\angle BCA \approx 32.6°$.

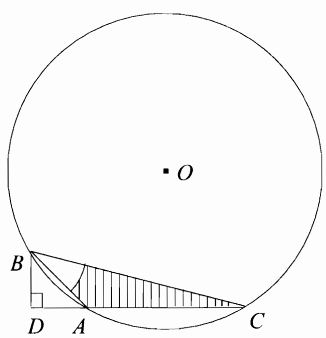


Figure 1

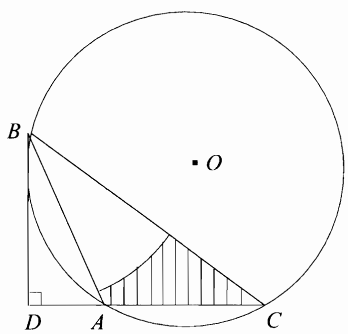


Figure 2

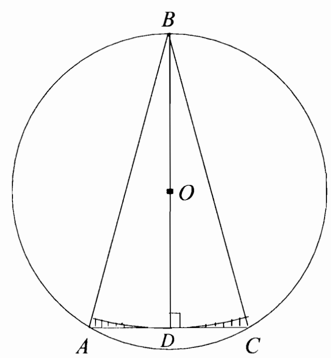


Figure 3

I–3. After the nth block of 3 wins, the lowest percentage occurs at the end of the next losing streak, i.e., after $1+2+\ldots+n=\frac{n(n+1)}{n}$ losses. Thus, $\frac{\text{wins}}{\text{total games}}$ equals $\frac{3n}{3n+\frac{n(n+1)}{2}}=\frac{1}{4} \rightarrow n=17$. After 51 wins and $51+\frac{17\cdot 18}{2}=204$ games, the winning percentage is 25%. Let x be the number of losses after the next block of 3 wins; we want the least x such that $\frac{51+3}{204+3+x}<\frac{1}{4}$. Solving, $x>9 \rightarrow x=10 \rightarrow$ $\boxed{217}$ games.

I–4. If $x_3=x_4$, then $f(x_3)=x_3 \rightarrow 4x_3-x_3^2=x_3 \rightarrow x_3=0$ or 3. If $f(x_2)=x_3$, then $4x_2-x_2^2=0$ or $3 \rightarrow x_2=0$ and 4 or $x_2=1$ and 3. Reject 0 and 3 since in those cases $x_2=x_3$. If $x_2=4$, then $f(x_1)=x_2 \rightarrow 4x_1-x_1^2=4 \rightarrow x_1=2$. If $x_2=1$, then $4x_1-x_1^2=1 \rightarrow x_1=2\pm\sqrt{3}$. In sum, $x_1=2$ generates 2, 4, 0, 0, . . . and $x_1=2\pm\sqrt{3}$ generates $2\pm\sqrt{3}$, 1, 3, 3, . . Thus, $x_1=2$ or $2\pm\sqrt{3}$ and the answer is $\boxed{3}$.

I–5. If x is the first term and y is the common difference, then $x+(x+y)+\ldots+(x+1998y)=1$ implies $\frac{1999}{2}(2x+1998y)=1 \rightarrow x+999y=\frac{1}{1999}$. If graphed, the interval from the origin to the y-intercept contains all common differences. Its width is $\frac{1}{999\cdot 1999}=\frac{1}{(1000-1)(2000-1)}=\boxed{\frac{1}{1997001}}$.

I–6. Let $N=\tan^{1997}\left(\frac{\pi}{12}\right)+\cot^{1997}\left(\frac{\pi}{12}\right)$. Since $\tan\frac{\pi}{12}=\frac{\tan\frac{\pi}{3}-\tan\frac{\pi}{4}}{1+\left(\tan\frac{\pi}{3}\cdot\tan\frac{\pi}{4}\right)}=\frac{\sqrt{3}-1}{1+\sqrt{3}}=2-\sqrt{3}$, then $\cot\frac{\pi}{12}=2+\sqrt{3}$. Thus, $N=\left(2-\sqrt{3}\right)^{1997}+\left(2+\sqrt{3}\right)^{1997}$. Looking at $\left(2-\sqrt{3}\right)^n+\left(2+\sqrt{3}\right)^n \pmod 9$ for $n=1, 2, 3, \ldots$ we obtain a repeating sequence of remainders of period 6: 4, 5, 7, 5, 4, 2, 4, 5, 7, 5, 4, 2, . . Since $1997 \equiv 5 \pmod 6$, the answer should be the 5th remainder, namely 4. To prove this use the Binomial Theorem:

$$\left(2-\sqrt{3}\right)^{1997} = 2^{1997} - {}_{1997}C_1 \cdot 2^{1996} \cdot \sqrt{3} + {}_{1997}C_2 \cdot 2^{1995} \cdot 3 - {}_{1997}C_3 \cdot 2^{1994} \cdot 3\sqrt{3} + {}_{1997}C_4 \cdot 2^{1993} \cdot 9 - \ldots$$

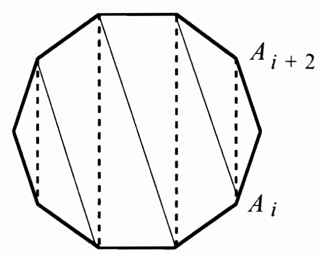
$$\left(2+\sqrt{3}\right)^{1997} = 2^{1997} + {}_{1997}C_1 \cdot 2^{1996} \cdot \sqrt{3} + {}_{1997}C_2 \cdot 2^{1995} \cdot 3 + {}_{1997}C_3 \cdot 2^{1994} \cdot 3\sqrt{3} + {}_{1997}C_4 \cdot 2^{1993} \cdot 9 + \ldots$$

Note that in N all terms with $\sqrt{3}$ will cancel and all terms with coefficients ${}_{1997}C_{2n}$ for $n \geq 2$ are divisible by 9. Thus $N \equiv 2\left(2^{1997} + \dfrac{1997 \cdot 1996}{2} \cdot 2^{1995} \cdot 3\right) \pmod 9$ which is congruent to $\left(2^{1998} + 1997 \cdot 998 \cdot 2^{1996} \cdot 3\right) \pmod 9$. Since $2^6 \equiv 1 \pmod 9$, then $2^{1998} = \left(2^6\right)^{332} \equiv 1 \pmod 9$

Also, $1997 \equiv 8 \pmod 9$, $998 \equiv 8 \pmod 9$, and $2^{1996} = \left(2^6\right)^{331} \cdot 2^4 \equiv 2^4 \pmod 9 = 7 \pmod 9$.

Thus $N \equiv 1 + 8 \cdot 8 \cdot 7 \cdot 3 \pmod 9 \equiv 4 \pmod 9$. The answer is $\boxed{4}$.

I–7. There are two types of parallels, those parallel to an edge and the dashed ones parallel to a segment connecting A_i and A_{i+2}. In the diagram there are 3 solid parallels making ${}_3C_2 = 3$ pairs of parallels and 4 dashed parallels making ${}_4C_2 = 6$ pairs. There are 5 distinct cases making the number of pairs of parallels equal to $5(6+3) = \boxed{45}$.



I–8. Let $t = \sqrt[3]{2}$ so that $(t+1)^3 = t^3 + 3t^2 + 3t + 1 = 2 + 3t^2 + 3t + 1 = 3\left(t^2 + t + 1\right)$. Multiply by $t - 1$:

$\left(t+1\right)^3\left(t-1\right) = 3\left(t^2 + t + 1\right)\left(t-1\right) = 3\left(t^3 - 1\right) = 3\left(2-1\right) = 3$. Thus $t - 1 = \dfrac{3}{(t+1)^3}$,

making $\sqrt[3]{t-1} = \dfrac{\sqrt[3]{3}}{t+1}$. Rationalize the denominator: $\dfrac{\sqrt[3]{3}}{t+1} \cdot \dfrac{t^2 - t + 1}{t^2 - t + 1} = \dfrac{\sqrt[3]{3}\left(t^2 - t + 1\right)}{t^3 + 1} =$

$\dfrac{\sqrt[3]{3}\left(t^2 - t + 1\right)}{3} = \dfrac{t^2 - t + 1}{\sqrt[3]{9}}$. Replacing t by $\sqrt[3]{2}$ gives $\dfrac{\sqrt[3]{2^2} - \sqrt[3]{2} + 1}{\sqrt[3]{9}} = \sqrt[3]{\dfrac{4}{9}} - \sqrt[3]{\dfrac{2}{9}} + \sqrt[3]{\dfrac{1}{9}}$.

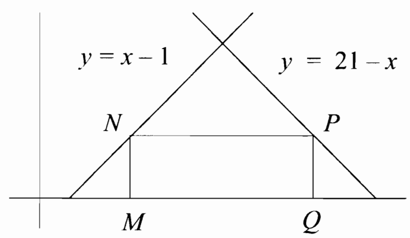
Hence, $a + b + c = \dfrac{4}{9} - \dfrac{2}{9} + \dfrac{1}{9} = \boxed{\dfrac{1}{3}}$.

ARML Relay #1 – 1997

R1–1. The complete graph of $|x|+|2y|=1$ is shown on the viewing window of a graphing calculator. When the region enclosed by the equation is measured by a ruler, its area is 120 square units. If the Xmin and Xmax of the viewing window are tripled, the Ymin and Ymax are doubled, and the new graph of the equation is measured by the same ruler, the area of the region is K square units. Compute K.

R1–2. Let T = TNYWR. If $K=\frac{T}{4}$, compute the number of real values of x for which the following equation is true: $(\log x)^{(K+\log x)}=1$.

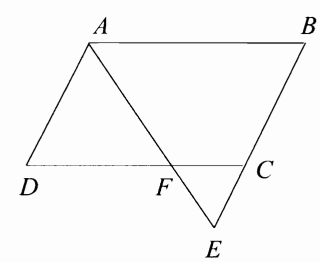
R1–3. Let T= TNYWR. Point N lies on $y=x-1$ and P lies on $y=21-x$ as shown. If the coordinates of M are $(T, 0)$ and $MNPQ$ is a rectangle, compute the area of $MNPQ$.



ARML Relay #2 – 1997

R2–1. Compute the number of integer values of n such that n^3+n^2+n+6 is divisible by $n+1$.

R2–2. Let T = TNYWR. In parallelogram $ABCD$, $\overline{EC}$ is the extension of $\overline{BC}$. If the area of ΔADF is 81 and the area of $\Delta FCE = T$, compute the area of $ABCD$.



R2–3. Let T = TNYWR and set $K=\frac{T}{2}$. If $y=x$ is tangent to $y=x^2+Kx+N$, then compute N.

Relay 1:

R1–1. 20

R1–2. 3

R1–3. 32

Relay 2:

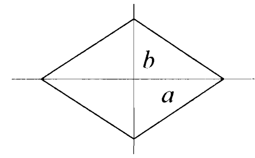
R2–1. 4

R2–2. 198

R2–3. 2401

Solutions to ARML Relay #1 – 1997

R1–1. From the diagram, $120 = 2ab$. Multiplying Xmin and Xmax by k multiplies the measured length by $\frac{1}{k}$. Thus, $a \to \frac{a}{3}$, $b \to \frac{b}{2}$ and the measured area $= 2\left(\frac{a}{3}\right)\left(\frac{b}{2}\right) = \left(\frac{1}{6}\right)(2ab) = \frac{1}{6} \cdot 120 = \boxed{20}$.



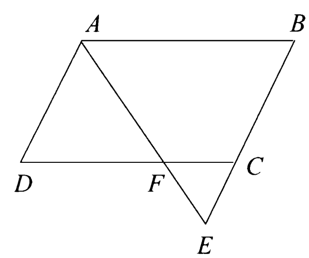
R1–2. We have $T = 20$ so $K = 5$. Regardless of K, $x = 10$ is a solution. If $\log x = -K$, then $(\log x)^{(K + \log x)} = (\log x)^0 = 1$, so $x = 10^{-K}$ is a solution. If K is odd and $\log x = -1$, i.e., $x = \frac{1}{10}$, then we have $(-1)^{2n} = 1$. Thus, there are 2 or 3 solutions depending on whether K is even or odd. Since K is odd, pass back $\boxed{3}$.

R1–3. We have $T = 3$. Since $NM = PQ = T - 1$, then $21 - x = T - 1 \to x = 22 - T$, giving $Q(22 - T, 0)$. Thus $MQ = (22 - T) - T = 22 - 2T$. Area $NPQM = 2(T - 1)(11 - T) = 2(2)(8) = \boxed{32}$.

Solutions to ARML Relay #2 – 1997

R2–1. $\dfrac{n^3 + n^2 + n + 6}{n+1} = \dfrac{n^2(n + 1) + (n + 1) + 5}{n+1} = n^2 + 1 + \dfrac{5}{n+1}$. Hence, $n + 1$ divides 5, so $n + 1 = \pm 1$ or $\pm 5 \to n = -6, -2, 0$, and 4, yielding $\boxed{4}$ solutions.

R2–2. Let $DF = a$, $FC = b$, and the heights of ΔADF and ΔECF be ha and hb. Since $a(\Delta ADF) = \frac{1}{2}ha^2 = 81$ and $a(\Delta ECF) = \frac{1}{2}hb^2 = T$, then $\left(ha^2\right)\left(hb^2\right) = (2 \cdot 81)(2T)$ giving $hab = 18\sqrt{T}$. Since $a(ABCD)$ equals $(ha)(a + b) = ha^2 + hab = 162 + 18\sqrt{T}$, then $a(ABCD) = 162 + 18\sqrt{4} = \boxed{198}$.



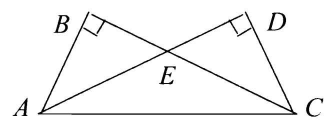
R2–3. We have $T = 198$, so $K = 99$. Set $x^2 + Kx + N = x \to x^2 + (K - 1)x + N = 0$.

For the line to be tangent, this equation must have one solution, so $(K - 1)^2 - 4N = 0 \to$

$$N = \left(\frac{K-1}{2}\right)^2 = 49^2 = \boxed{2401}.$$

Note: Pass answers from position 1 to position 15.

1. If $AB = DC = 24$ and $BC = AD = 32$, compute the area of ΔAEC.

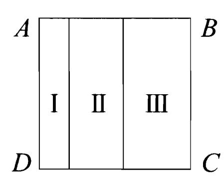


2. Let T = TNYWR and set $K = \frac{T}{100}$. If $\log_2 b - \log_2 a = K$, then $b^2 - a^2 = Ma^2$. Compute M.

3. Let T = TNYWR and set $K = \frac{T}{21}$. Compute the largest angle θ in $[0°, 360°)$ such that $\cos(K\theta) = \frac{1}{2}$.

4. Let T = TNYWR. Let $f(g(x)) = Tx + T$ and $f(x) = x + 2T$. If $g(x) = ax + b$, compute $a + b$.

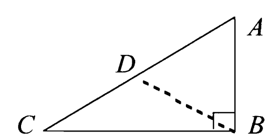
5. Let T = TNYWR. If the side of square $ABCD = T + 9$ and the areas of regions I, II, and III are in increasing arithmetic progression, compute the area of region II.



6. Let T = TNYWR and set $K = \frac{T}{9}$. If $\frac{1 + 7i}{4 + Ki} = a + bi$, compute $a + b$.

7. Let T = TNYWR. If $ab = T$, $bc = \frac{1}{T}$, and $ac = T^2$, compute $|a + c|$.

8. Let T = TNYWR. If $AB = 3T$, $BC = 4T$, $AC = 5T$, and $AD = 3T$, compute the area of ΔADB.



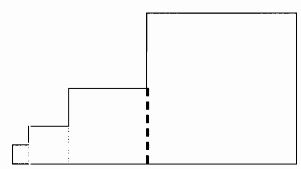
9. Let T = TNYWR and set $K = \frac{T}{30}$. If $9 \cdot K + AB = BA$ for two-digit base ten numbers AB and BA, compute the largest possible value for the two-digit number BA.

10. Let T = TNYWR. If $f(x) = \frac{5x}{T}$ is defined only on $0 \le x \le T^2$, compute the number of lattice points on f.

11. Let T = TNYWR and set $K = \frac{T-6}{7}$. Given isosceles ΔABC with $m\angle B = 90°$ and $AB = K$, circle O is tangent to $\overline{AC}$ at D, the midpoint of $\overline{AC}$, and passes through B. If the diameter of O is $\frac{a}{\sqrt{b}}$, compute $a + b$.

12. Let T = TNYWR. If $x - \frac{x-T}{4} = \frac{T}{3} + \frac{T-3x}{6}$, compute x.

13. Let T = TNYWR and set $K = 40T - 5$. Each figure is a square with a side half as long as the square to its right. If the perimeter of the whole figure is K, compute the area of the smallest square.

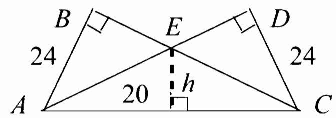


14. Let T = TNYWR. A retail store pays a wholesale price of \$$T$ per toy. It prices the toy at \$$K$ so that at a 25% off sale, the store still makes a profit of 20%. Compute K.

15. Let T = TNYWR. A linear function f is such that $f(0) = T$ and $f(2x) = 2f(x-1)$ for all x. Compute the area of the triangle formed by the intercepts of f and the origin.

1. 300
2. 63
3. 340°
4. 0
5. 27
6. 2
7. 5
8. 90
9. 96
10. 97
11. 15
12. 3
13. $\frac{25}{4}$
14. 10
15. 10

1. ΔABC and ΔCDA are 3-4-5 right triangles, so $AC = 40$, making $\dfrac{h}{20} = \dfrac{3}{4} \to h = 15 \to \text{area} = \dfrac{1}{2} \cdot 40 \cdot 15 = \boxed{300}$.



2. $\log_2 \dfrac{b}{a} = K \to \dfrac{b}{a} = 2^K \to b^2 = a^2 \cdot 2^{2K}$. Since $K = 3$, $b^2 = 64a^2 \to b^2 - a^2 = 63a^2 \to M = \boxed{63}$.

3. $\text{Cos } K\theta = \dfrac{1}{2} \to K\theta = 60° + 360n$ or $300° + 360n$. Since $K = 3$, $\theta = 20° + 120n$ or $100° + 120n$. Thus, the largest occurs when $n = 2$, and that makes $\theta = \boxed{340°}$.

4. $f\big(g(x)\big) = g(x) + 2T = Tx + T \to g(x) = Tx - T$. Thus, regardless of T, $a + b = \boxed{0}$.

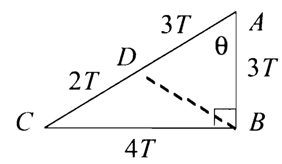
5. $\text{II} - d + \text{II} + \text{II} + d = (T+9)^2 \to \text{II} = \dfrac{(T+9)^2}{3}$. Since $T = 0$, $\text{II} = \boxed{27}$.

6 $\dfrac{1+7i}{4+Ki} \cdot \dfrac{4-Ki}{4-Ki} = \dfrac{(4+7K)+(28-K)i}{16+K^2} \to a + b = \dfrac{32+6K}{16+K^2}$. Since $K = 3$, $a + b = \dfrac{50}{25} = \boxed{2}$.

7. $ab \cdot ac \cdot bc = (abc)^2 = T \cdot \dfrac{1}{T} \cdot T^2 = T^2 \to abc = \pm T$. Since $ab = T$, then $c = \pm 1$. If $c = 1$, then $a = T^2$. If $c = -1$, then $a = -T^2$. Since $\left| -T^2 - 1 \right| = \left| T^2 + 1 \right|$, $|a + c|$ is fixed. Since $T = 2$, $|a + c| = 2^2 + 1 = \boxed{5}$.

8. $\sin\theta = \frac{4}{5} \rightarrow$ area $\Delta ADB = \frac{1}{2} \cdot 3T \cdot 3T \cdot \sin\theta = \frac{9T^2}{2} \cdot \frac{4}{5} = \frac{18T^2}{5}$.

Since $T = 5$, the area $\Delta ADB = \boxed{90}$.



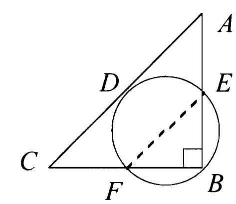
9. Before receiving T, try values for K: $K = 1 \rightarrow 89 + 9 = 98$, $K = 2 \rightarrow 79 + 18 = 97$, $K = 3 \rightarrow 69 + 27 = 96, \ldots$ Since $K = 3$, the largest value for BA is $\boxed{96}$.

10. If 5 and T are relatively prime, then y is integral when x is a multiple of T. On $\left[0, T^2\right]$, $x = 0, T, 2T, \ldots, T \cdot T$, making $T + 1$ values for x. If T and 5 are not relatively prime, be patient. Since $T = 96$, the answer is $\boxed{97}$.

11. Since $AE \cdot AB = AD^2$ and $AD = \frac{AB\sqrt{2}}{2}$, then $AE \cdot AB = \frac{AB^2}{2} \rightarrow$

$AE = \frac{AB}{2}$, so the diameter $\overline{EF}$ of the circle is also the midline of

$\Delta ABC \rightarrow EF = \frac{K\sqrt{2}}{2}$. Since $K = 13$, $EF = \frac{13}{\sqrt{2}}$. The sum $= \boxed{15}$.



12. $12x - 3(x - T) = 4T + 2(T - 3x) \rightarrow 9x + 3T = 6T - 6x \rightarrow x = \frac{T}{5}$. Since $T = 15$, $x = \boxed{3}$.

13. $4x + 3(2x + 4x + 8x) = 46x = K \rightarrow$ area $= \left(\frac{K}{46}\right)^2$. Since $T = 3$, $K = 115$, and the area $= \boxed{\frac{25}{4}}$.

14. $\frac{3K}{4} = \frac{6T}{5} \rightarrow K = \frac{8T}{5} = \frac{8}{5} \cdot \frac{24}{5} = \boxed{10}$.

15. If $x = 1$, $f(2) = 2f(0) = 2 \cdot T$. Thus, points $(0, T)$ and $(2, 2T)$ lie on the line. The equation of the line is $y = \frac{Tx}{2} + T$ and its intercepts are $(0, T)$ and $(-2, 0)$. Area $= \frac{1}{2} \cdot 2 \cdot T = T = \boxed{10}$.

1. Let $S_n = \left[\sqrt{1}\right] + \left[\sqrt{2}\right] + \left[\sqrt{3}\right] + \ldots + \left[\sqrt{n}\right]$ where $[x]$ is the greatest integer function.

   Compute the largest value of $k < 1997$ such that $S_{1997} - S_k$ is a perfect square.

1. For $1936 \le n \le 1997$, $\left[\sqrt{n}\right] = 44$. So, for $1936 \le k < 1997$, $S_{1997} - S_k = 44(1997 - k) = 4 \cdot 11 \cdot (1997 - k)$. For the difference to be a perfect square, $(1997 - k)$ must equal 11, giving $1997 - k = 11 \rightarrow k = \boxed{1986}$.

ARML

1998

ARML continued to grow, continuing to establish itself as an intellectually challenging and thoroughly enjoyable mathematics competition. This year there were 27 teams in Division A, 75 teams in Division B and 9 alternate teams for a total of 111 teams fielding upwards of 1680 students. Competition was very spirited with only 4 points separating the top two teams in Division A. This year a team from Taiwan took part in the competition for the first time and this led to an invitation for Mark Saul, president of ARML, to travel to Taiwan to meet with educators to explore further ways in which Taiwan could participate in ARML.

Ted Alper and Bill Kling received the Samuel Greitzer Distinguished Coach Award.

Ted was a member of the Montgomery County ARML team in 1979 and 1980. In 1994 while working for the Education Program for Gifted Youth at Stanford, he began recruiting students in the San Francisco Bay area to form ARML teams. His teams quickly became very successful and in 1996, the San Francisco Bay Area team won the A Division.

Bill has started math teams throughout his career. In 1982 he organized the first Upstate New York ARML team and shepherded it through the next decade, eventually creating three different ARML teams. This year was the last year that he planned on being involved with the team.

Matteo Paris received the Alfred Kalfus Founder's Award. While a junior in high school in California in 1989, Matteo first heard about ARML and was determined to create a team. Then the only site was at Penn State, but Matteo and Jeff Wall, another high school senior, organized, recruited, and trained a team that eventually got to Penn State and finished in the top ten. Since then Matteo has organized teams, become an Executive Board member as well as a coordinator for the UNLV site. While a student he also won the Zachary Sobol Award.

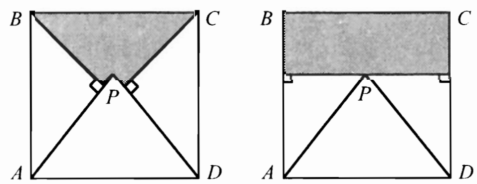
Chris Clark of Western Massachusetts and Greg Tseng of Thomas Jefferson HS received the Zachary Sobol Awards.

T–1. In acute-angled triangle ABC, $m\angle A = (x + 15)°$, $m\angle B = (2x - 6)°$, and the exterior angle at C has measure $(3x + 9)°$. Compute the number of possible integral values of x.

T–2. Two of the vertices of a square are $A(\log_{15}5, 0)$ and $B(0, \log_{15}x)$ for $x > 1$. The other two vertices lie in the first quadrant. Add the coordinates of all four vertices. The result is 8. Compute x.

T–3. If $a_1 = a_2 = 1$ and $a_{n+2} = \dfrac{a_{n+1} + 1}{a_n}$ for $n \geq 1$, compute a_t where $t = 1998^5$.

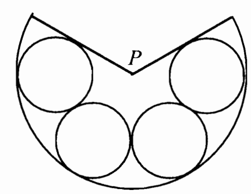
T–4. Point P lies in a square $ABCD$ of side 2 so that $PA = PD$. If the two shaded regions have equal areas, compute $\cos \angle PAD$.



T–5. Line ℓ is tangent to the circle $x^2 + y^2 = 1998$ at $T(a, b)$ in the first quadrant. If the intercepts of ℓ and the origin form the vertices of a triangle whose area is 1998, compute the product $a \cdot b$.

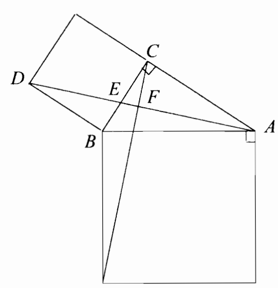
T–6. Compute all ordered pairs (x, y) such that: $\begin{cases} xy + 9 = y^2 \\ xy + 7 = x^2 \end{cases}$

T–7. Four congruent circles are tangent to each other and tangent to the edges of a sector as shown. If the straight edges are joined to form a right circular cone with vertex at P, the radius of the base would be 2/3 the slant height of the cone. Compute the ratio of the radius of the sector to the radius of each circle.



T–8. Let S be the set of lattice points in the region defined by $0 \leq x \leq 3$ and $0 \leq y \leq 3$. Triangles are formed by choosing three non-collinear members of S as vertices. Two triangles are distinct if they share no more than two vertices. Compute the number of distinct triangles with an area of 3/2.

T–9. Squares are erected on the sides of right triangle ABC as shown. If $BC = 68$ and $AC = 204$, compute the absolute value of the difference between the areas of ΔBDE and ΔCEF.



T–10. Let f be a function whose domain is $S = \{1, 2, 3, 4, 5, 6\}$ and whose range is contained in S. Compute the number of different functions f which have the following property: no range value y comes from more than three arguments x in the domain. For example, $f = \{(1,1), (2,1), (3,1), (4,4), (5,4), (6,6)\}$ has the property, but $g = \{(1,1), (2,1), (3,1), (4,1), (5,3), (6,6)\}$ does not.

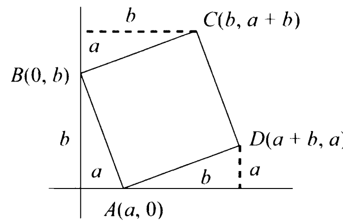
1. 20
2. 45
3. 2
4. $\frac{1}{2}$
5. 999
6. $\left(\frac{7}{4},-\frac{9}{4}\right),\left(-\frac{7}{4},\frac{9}{4}\right)$
7. 3
8. 92
9. 272
10. 44,220

T–1. Since an exterior angle of a triangle is equal to the sum of the two remote interior angles, we set $3x + 9 = (x + 15) + (2x - 6)$, but that is an identity and is of no help. Since the triangle is acute-angled, we have several conditions to consider. First, $m\angle A < 90° \rightarrow x + 15° < 90° \rightarrow x < 75°$. Second, $m\angle B < 90° \rightarrow 2x - 6 < 90° \rightarrow x < 48°$. Third, since $\angle C$ is acute, then $3x + 9°$ is obtuse, giving $90° < 3x + 9° < 180° \rightarrow 27° < x < 57°$. Considering all the conditions, we have $27° < x < 48°$, giving $x \in \{28°, 29°, \ldots, 46°, 47°\}$. Thus, the number of integral values of x is $\boxed{20}$.

T–2. The sum of the coordinates of the vertices is clearly

$4a + 4b = 4(\log_{15} 5 + \log_{15} x) = 8$. Thus,

$\log_{15}(5x) = 2 \rightarrow 5x = 15^2 \rightarrow x = \boxed{45}$.



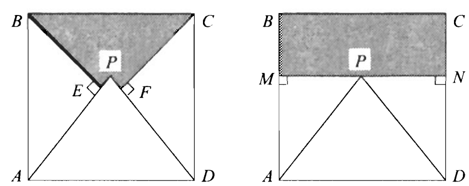
T–3. Since $a_1 = 1$, $a_2 = 1$, $a_3 = \dfrac{1+1}{1} = 2$, $a_4 = \dfrac{2+1}{1} = 3$, $a_5 = \dfrac{3+1}{2} = 2$, $a_6 = \dfrac{2+1}{3} = 1$ and

$a_7 = \dfrac{1+1}{2} = 1$, the sequence is 1, 1, 2, 3, 2, 1, 1, 2, 3, 2, . . . and is clearly cyclic with a period of 5.

Since $1998 = 3 \pmod 5$, $1998^5 = 3^5 \pmod 5 = 3 \pmod 5$ and for $t = 1998^5$, a_t is the third element in the sequence, namely $\boxed{2}$.

T–4. Clearly, $\Delta BEA \cong \Delta CFD$ and $\Delta PMA \cong \Delta PND$. Since the shaded regions have equal areas, all four triangles are congruent, making $AB = PA$, so ΔPAD is equilateral

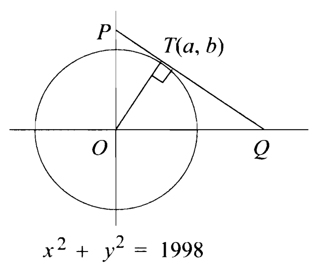
giving $\cos \angle PAD = \cos 60° = \boxed{\dfrac{1}{2}}$.



T–5. Since the slope of $\overline{OT} = \dfrac{b}{a}$, then the slope of $\overline{PQ} = -\dfrac{a}{b}$, making the equation of $\overline{PQ}$ equal $y - b = \dfrac{-a}{b}(x - a)$. If $x = 0$, then $y - b = \dfrac{a^2}{b}$

$\rightarrow y = \dfrac{a^2 + b^2}{b} = \dfrac{1998}{b}$. If $y = 0$, then $-b = \dfrac{-a}{b}(x - a)$ giving

$x = \dfrac{b^2}{a} + a = \dfrac{a^2 + b^2}{a} = \dfrac{1998}{a}$.



The area of $\Delta POQ = 1998 = \dfrac{1}{2}\left(\dfrac{1998}{a}\right)\left(\dfrac{1998}{b}\right) \rightarrow$ the product $a \cdot b = \boxed{999}$.

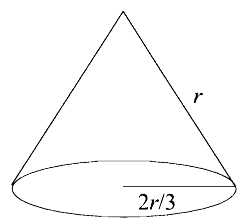
T–6. Subtracting the bottom equation from the top yields $y^2 - x^2 = 2 \rightarrow (x - y)(x + y) = -2$. Adding the two equations yields $2xy + 16 = x^2 + y^2 \rightarrow x^2 - 2xy + y^2 = 16 \rightarrow (x - y)^2 = 16 \rightarrow x - y = \pm 4$.

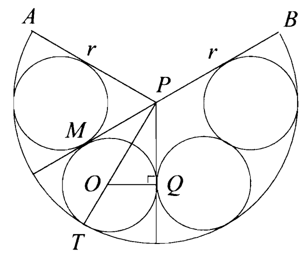
If $x - y = 4$, then $4(x + y) = -2 \rightarrow x + y = -\frac{1}{2}$. Adding $x - y = 4$ and $x + y = -\frac{1}{2}$, we obtain $x = \frac{7}{4}$ and $-\frac{9}{4}$. If $x - y = -4$, we obtain $x = -\frac{7}{4}, y = \frac{9}{4}$ in similar fashion.

Thus, $(x, y) = \boxed{\left(\frac{7}{4}, -\frac{9}{4}\right), \left(-\frac{7}{4}, \frac{9}{4}\right)}$.

T–7. The perimeter of the cone's base $= 2\pi\left(\frac{2r}{3}\right) = \frac{4\pi r}{3}$.

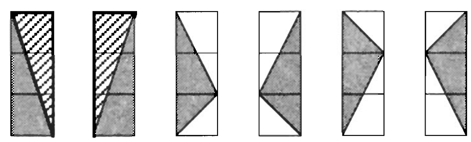
The circumference of the circle containing the sector is $2\pi r$, so $m\overset{\frown}{ATB} = 240°$. Thus tangents to each circle form a 60° angle, making $PO = 2(OQ)$.



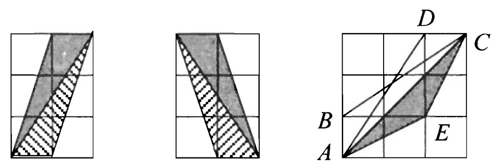


Since $PT = r = OQ + 2 \cdot OQ = 3 \cdot OQ$, then $\frac{r}{OQ} = \boxed{3}$.

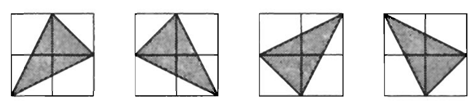
T–8 From each vertical or horizontal strip of three squares, 8 triangles with area 3/2 can be formed, giving 8 x 3 x 2 = 48 triangles.



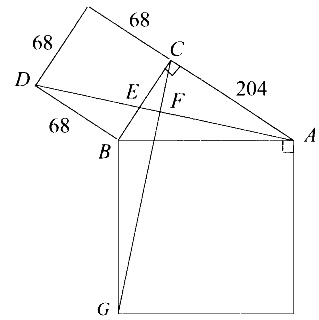
From each vertical or horizontal strip of 6 squares, 4 triangles can be formed. There are 2 such vertical and horizontal strips for a total of 4 x 2 x 2 = 16 triangles.



On each diagonal of the 3 x 3 square 6 triangles can be formed, 3 above and 3 below. For example, ΔABC, ΔAEC, and ΔADC.



Four triangles can be put in each 2 x 2 square. Since there are 4 such 2 x 2 squares, there are 4 x 4 = 16 triangles. Total: 48 + 16 + 12 + 16 = $\boxed{92}$.



T–9. From $\overline{DB} \parallel \overline{AC}$ we obtain $\angle BDE \cong \angle EAC$ and since $\angle DBE \cong \angle ACE$, then $\Delta DBE \sim \Delta ACE$. Thus, $\dfrac{DB}{BE} = \dfrac{AC}{CE}$. Since $AC = 204$, $DB = BC = 68$, and $CE = BC - BE = 68 - BE$, we have $\dfrac{68}{BE} = \dfrac{204}{68 - BE} \rightarrow BE = 17$ and $CE = 51$. Then $DE = \sqrt{(4 \cdot 17)^2 + 17^2} = 17\sqrt{17}$. $\Delta DBA \cong \Delta CBG$ by SAS so $\angle EDB \cong \angle ECF$ and since $\angle BED \cong \angle FEC$, then $\Delta DBE \sim \Delta CFE$ by AA. Thus, $\dfrac{DE}{BE} = \dfrac{CE}{FE} \rightarrow$ $\dfrac{17\sqrt{17}}{17} = \dfrac{51}{EF} \rightarrow EF = \dfrac{51}{\sqrt{17}}$ making $CF = \dfrac{4 \cdot 51}{\sqrt{17}}$. The difference in areas equals $\dfrac{1}{2}\left(68 \cdot 17 - \dfrac{51}{\sqrt{17}} \cdot \dfrac{4 \cdot 51}{\sqrt{17}}\right) = 16 \cdot 17 = \boxed{272}$. It can be shown in all cases that the difference in areas is equal to the square of the distance from B to $\overline{DE}$.

T–10. The sizes of the preimages under f of $\{1, \ldots, 6\}$ must partition 6. The excluded partitions are those containing a 4 or greater, that is 4–1–1, 4–2, 5–1, and 6. There are $6 \cdot 5 \cdot 4 \cdot \left({}_6C_4\right) = 1800$ functions of the 4–1–1 type since there are 6 ways to pick a range element that four domain elements map to, 5 • 4 ways to pick a range element that one domain element maps to, and ${}_6C_4$ ways to pick four elements in the domain to map to the single range value. Similarly, there are $6 \cdot 5 \cdot \left({}_6C_4\right) = 450$ ways to get preimages of size 4–2, $6 \cdot 5 \cdot \left({}_6C_5\right) = 180$ ways to get 5–1, and $6\left({}_6C_6\right) = 6$ ways to get 6. Thus, there are $1800 + 450 + 180 + 6 = 2436$ excluded functions. The total number of mappings from S to S is $6^6 = 46656$, so the number of functions with the desired property are $46656 - 2436 = \boxed{44220}$.

For each of the following problems write complete solutions. Justify each answer. Angles are in degrees.

1. Let positive integers A, B, and C be the angles of a triangle such that $A \le B \le C$.

   a) Determine all the values that each of A, B, and C can take on.

   b) Compute the number of ordered triples (A, B, C) in which $B = 70°$.

2. In convex pentagon $ABCDE$, $m\angle A < m\angle B < m\angle C < m\angle D < m\angle E$. Let $T = m\angle C + m\angle D$.

   If $m\angle A : m\angle B : m\angle C : m\angle D : m\angle E = 1:2:x:y:5$, determine the range of values of T.

3. Let a, b, and c be positive integers such that $a < 3b$, $b > 4c$, and $a + b + c = 200$.

   a) Determine the largest value that c can take on.

   b) Determine the smallest value that b can take on.

   c) Determine the number of ordered triples (a, b, c) in which $c = 11$.

4. Let a, b, and c be positive integers. If $a + b + c = 85$, $c > 3a$, $2b > c$, and $5a > 3b$,

   prove algebraically that there is a unique solution (a, b, c) to this system.

5. A unit square is divided into 4 rectangles of positive area by two cuts parallel to the sides of the square.

   Let $a_1 \le a_2 \le a_3 \le a_4$ be the areas of the four parts in non-decreasing order. For each

   $i = 1, \ldots, 4$, determine with proof the range of values for a_i.

6. A unit cube is divided into 8 parallelepipeds of positive volume by three cuts parallel to the faces of the cube.

   Let $v_1 \le v_2 \le \ldots \le v_8$ be the volumes of the eight parts in non-decreasing order. Determine with proof the

   range of values for v_4 and v_5.

7. Let n be a positive integer. Allie and Bob play a game constructing a partition $n = a_1 + a_2 + \ldots + a_k$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$. Allie wins if there is an odd number of terms in the partition, i.e., if k is odd; Bob wins otherwise. Allie begins by choosing a number a_1 between 1 and $n - 1$ inclusive. Bob then chooses a number a_2 between a_1 and 1 inclusive such that $a_1 + a_2 \leq n$. Allie then chooses an a_3 between a_2 and 1 inclusive such that $a_1 + a_2 + a_3 \leq n$, and so on, with the game ending when the partition is complete. Determine with proof all $n > 1$ for which Bob has a winning strategy.

8. Allie and Bob play a game similar to the one in #7 except that the inequality $a_i \geq a_{i+1}$ is replaced by $2a_i \geq a_{i+1}$. Prove that Bob has a winning strategy if and only if n is a Fibonacci number. (You may assume the following: each positive integer n can be uniquely represented as a decreasing sum of non-adjacent Fibonacci numbers, i.e., $32 = 21 + 8 + 3$.)

1a. The maximum of C occurs when $A = B = 1° \rightarrow C = 178°$. The minimum of C occurs when $A = B = C = 60°$. The maximum of B occurs if $A = 1°$ and B is as close to C as possible. Since $B + C = 179°$, $B = 89°$ and $C = 90°$. The maximum of A occurs when $A = B = C \rightarrow A = 60°$. Thus, $1° \leq A \leq 60°$, $1° \leq B \leq 89°$, $60° \leq C \leq 178°$.

1b. If $B = 70°$, then $A + C = 110°$ and $C \geq 70° \rightarrow A \leq 40°$. For $A = 1°$ to $40°$, $B = 70°$, and $C = 109°$ to $70°$ respectively. The number of ordered triples is 40.

2. For an upper bound of T consider an extreme case: let $x = y = 5$, giving $1k : 2k : 5k : 5k : 5k$ for the angles. If $18k = 540°$, then $k = 30° \rightarrow m\angle C + m\angle D < 150° + 150° = 300°$. For a lower bound of T, note that $5m\angle A = m\angle E < 180°$ so $m\angle A < 36°$, making $m\angle C + m\angle D = 540° - m\angle A - m\angle B - m\angle E = 540° - 8m\angle A > 540° - 8 \cdot 36° = 252°$. This bound is the best possible. Let ε be very small and let $(m\angle A, m\angle B, m\angle C, m\angle D, m\angle E) = (36° - \varepsilon, 72° - 2\varepsilon, 100°, 152° + 8\varepsilon, 180° - 5\varepsilon)$. These values satisfy the inequalities and $m\angle C + m\angle D$ is as close to $252°$ as we like. Thus, $252° < T < 300°$.

3a. If $a \geq 1$, then $b + c \leq 199$ and since $b > 4c$, then $4c + c < 199 \rightarrow c < 39.8 \rightarrow$ the maximum of c is 39, achieved when $(a, b, c) = (1, 160, 39)$.

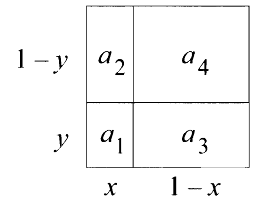
3b. Since $3b > a$, then $3b + b + c > 200 \rightarrow 4b + c > 200$. Since $b > 4c$, $\frac{b}{4} > c \rightarrow 4b + \frac{b}{4} > 200 \rightarrow b > 47.06 \rightarrow$ the minimum value of $b = 48$, achieved when $(a, b, c) = (141, 48, 11)$.

3c. If $c = 11$, then $a + b = 189$ and since $b \geq 48$, then $a \leq 141$. All such (a, b, c) satisfy the two inequalities, so the ordered triples range from $(141, 48, 11)$ to $(1, 188, 11)$, a total of 141.

4. Given that $a + b + c = 85$ and $c > 3a$, then $\frac{c}{3} > a$ giving $\frac{c}{3} + b + c > 85$. Since $5a > 3b$, then $3a > \frac{9b}{5}$ giving $c > \frac{9b}{5} \rightarrow \frac{5c}{9} > b$. Thus, $\frac{c}{3} + \frac{5c}{9} + c > 85 \rightarrow c > 45$. Similarly, $2b > c$ implies $b > \frac{c}{2}$ giving $a + \frac{c}{2} + c < 85$. Also, $5a > 3b \rightarrow \frac{10a}{3} > 2b > c$, making $a > \frac{3c}{10}$. Thus, $\frac{3c}{10} + \frac{c}{2} + c < 85 \rightarrow c < 47.22 \rightarrow c = 46$ or 47. If $c = 46$, then $a + b = 39$ and $2b > c \rightarrow 2b > 46 \rightarrow b \geq 24$. Also, $c > 3a \rightarrow 46 > 3a \rightarrow a \leq 15$. Clearly, $(15, 24, 46)$ works, but if

$a \le 14$ making $b \ge 25$, $5a$ is not greater than $3b$. If $c = 47$, $2b > 47$ and $47 > 3a$ gives $b \ge 24$ and $a \le 15$, but since $a + b = 38$, $a \ne 15$ so $a \le 14$, but then $5a$ is not greater than $3b$. Hence, no solutions if $c = 47$. The only solution is $(15, 24, 46)$.

5. Let $0 < x \le y \le \frac{1}{2}$. This is equivalent to $x \le 1 - x$, $y \le 1 - y$, $x \le y$, and therefore $1 - y \le 1 - x$. Then $a_1 \le a_2 \le a_3 \le a_4$ can be written as $xy \le x(1 - y) \le y(1 - x) \le (1 - x)(1 - y)$. As x, y approach 0, the greatest lower bound of a_1, a_2, and a_3 is clearly 0, and the least upper bound of $a_4 = 1$.



Since $a_1 = \frac{1}{4}(a_1 + a_1 + a_1 + a_1) \le \frac{1}{4}(a_1 + a_2 + a_3 + a_4) = \frac{1}{4}$, then $a_1 \le \frac{1}{4}$ with equality when $x = y = \frac{1}{2}$. Now $a_3 = \frac{1}{2}(a_3 + a_3) \le \frac{1}{2}(a_3 + a_4) < \frac{1}{2}$. If $y = \frac{1}{2}$, and x approaches 0, then a_3 approaches, but does not equal $\frac{1}{2}$. Since $a_2 = x(1 - y) \le x(1 - x) = \frac{1}{4} - \left(\frac{1}{2} - x\right)^2 \le \frac{1}{4}$, then $a_2 \le \frac{1}{4}$ with equality when $x = y = \frac{1}{2}$. Finally, a_4 is minimized when x and y are as large as possible, i.e., $x = y = \frac{1}{2}$, making $a_4 = \frac{1}{4}$. Thus, $0 < a_1, a_2 \le \frac{1}{4}$, $0 < a_3 < \frac{1}{2}$, $\frac{1}{4} \le a_4 < 1$.

6. Method 1: Without loss of generality, take $x \le 1 - x$, $y \le 1 - y$, $z \le 1 - z$, and $x \le y \le z$ which implies that $1 - z \le 1 - y \le 1 - x$. Clearly $v_4 > 0$. To find the upper bound, divide the volumes into 4 pairs:

$$\{xyz, (1 - x)(1 - y)(1 - z)\} \qquad \{(1 - x)yz, x(1 - y)(1 - z)\}$$
$$\{x(1 - y)z, (1 - x)y(1 - z)\} \qquad \{xy(1 - z), (1 - x)(1 - y)z\}$$

Consider the lesser of each pair. Since v_4 is the fourth smallest volume it must be less than or equal to one such lesser pair member. But the product of the numbers in each pair is $x(1 - x)y(1 - y)z(1 - z)$. Since $A(1 - A) \le \frac{1}{4}$ for positive A, the product of each pair is at most $\frac{1}{64}$, which means that the lesser of each pair is at most $\frac{1}{8}$. Since, if $x = y = z = \frac{1}{2}$, then all eight volumes equal $\frac{1}{8}$, and the upper bound is attainable.

Thus, $0 < v_4 \le \frac{1}{8}$. For v_5, note that $4v_5 \le v_5 + v_6 + v_7 + v_8 < 1$, so $0 < v_5 < \frac{1}{4}$.

This solution cleverly avoids a thorny problem, addressed in method #2, that volumes $(1 - x)yz$ and

$x(1-y)(1-z)$ which correspond to v_4 and v_5 respectively, can switch order, i.e., if $x = \frac{1}{5}$ and $y = z = \frac{1}{2}$, we obtain $v_4 = \frac{1}{5}$ and $v_5 = \frac{1}{20}$, but if $x = z = \frac{1}{2}$ and $y = \frac{1}{10}$, we obtain $v_4 = \frac{1}{40}$ and $v_5 = \frac{9}{40}$.

Method #2: The volumes can be written as $v_1 = xyz$, $v_2 = xy(1-z)$, $v_3 = x(1-y)z$, $v_4 = \text{min of } (1-x)yz \text{ or } x(1-y)(1-z)$, $v_5 = \text{max of } (1-x)yz \text{ or } x(1-y)(1-z)$, $v_6 = (1-x)y(1-z)$, $v_7 = (1-x)(1-y)z$, and $v_8 = (1-x)(1-y)(1-z)$. Consider the product $v_4 v_5 = x(1-x)y(1-y)z(1-z)$. Since $A(1-A) \le \frac{1}{4}$, $v_4 v_5 \le \frac{1}{4^3}$, so the smaller, v_4, can be no larger than $\frac{1}{8}$, with equality at $x = y = z = \frac{1}{2}$. Since in either $(1-x)yz$ or $x(1-y)(1-z)$, one of the variables can go to 0, then $0 < v_4 \le \frac{1}{8}$. For v_5, if x, y, and z are very small, both $(1-x)yz$ and $x(1-y)(1-z)$ are very small, making 0 the lower bound. For the upper bound, note that

$$x(1-y)(1-z) \le x(1-x)(1-x) = \frac{1}{2}\left(2x(1-x)(1-x)\right) \le \frac{1}{2}\left(\frac{2x + (1-x) + (1-x)}{3}\right)^3$$

by the Arithmetic-Geometric Mean Inequality. The last expression equals $\frac{4}{27}$. Since $(1-x)yz < 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, then by $v_5 = \text{max of } (1-x)yz \text{ or } x(1-y)(1-z)$ we choose $v_5 < \frac{1}{4}$. This is the best possible bound since if $y = z = \frac{1}{2}$ and x approaches 0, then v_5 approaches, but does not equal $\frac{1}{4}$.

7. Claim: Bob has a winning strategy if and only if $n = 2^k$.
Proof: Consider this as a game of removing matches from a pile of n matches.

Observe what happens in simple cases:

1. If n is odd > 1, Allie has a perfect winning strategy. She takes 1 match, i.e., $a_1 = 1$, leaving an even number. Bob is forced to take 1 match, leaving an odd number, and this continues until Allie takes the last match.

2. If $n = 2$, Bob has a winning strategy. Allie must take 1 match and then Bob takes the other.

3. If n is even, greater than 2 and not a multiple of 4, Allie has a perfect winning strategy. She takes 2 matches, leaving a multiple of 4. Bob can take 1 or 2. If he takes 1, he loses by (1). If he takes 2, then $n-4$ matches are left. If, initially, $n = 6$, then there are two matches left and Allie can take them both.

Otherwise, $n - 4$ matches represent an even number, greater than 2, and not a multiple of 4, so Allie can repeat the process, eventually reaching 6.

4. If $n = 4$, Bob has a winning strategy. Allie will not take 1 or 3, she'll take 2, but then Bob takes 2 and wins.

Now let $p(q)$ represent the highest power of 2 dividing q. Thus, $p(17) = 2^0 = 1$, $p(24) = 8$. Consider the following <u>lemma</u>: if at a certain stage there are q matches left, and if the next person to play is allowed to take $p(q)$ matches, then that player can guarantee a win by removing $p(q)$ matches.

<u>Proof by induction</u>: The result holds for $q = 1$ since $p(1) = 2^0 = 1$. Assume it holds true for all positive integers less than q. Suppose it is X's turn. There are $q = 2^m \cdot s$ matches left for s an odd number, m an integer ≥ 0. X removes $p(q) = 2^m$ matches leaving $2^m \cdot s - 2^m$. If $q = p(q)$, i.e., if $s = 1$, then X has won. Otherwise, although Y is looking at a number of the form $2^m(s - 1)$ which is divisible by a higher power of 2 than was 2^m. Now Y isn't allowed to remove more than 2^m. Suppose Y removes $2^w \cdot t$ matches for t odd. Then there remain $2^m(s - 1) - 2^w \cdot t = 2^a \cdot b$ matches for b odd. Clearly, $w = a$, and so X is allowed to remove $p\left(2^a \cdot b\right) = 2^a$ matches, and thus, by the inductive hypothesis, X wins.

Applying this lemma to our problem: if n is not a power of 2, Allie can win if she plays perfectly by removing the highest power of 2 dividing n. If n is a power of 2, and Allie removes a certain number of matches, leaving q, then Bob is allowed to remove $p(q)$ matches, so Bob can force a win.

8. The following example may help: Let $n = 13$. Suppose Allie takes 2 leaving $11 = 8 + 3$. Bob takes away 3, leaving 8. If Allie takes away more than 2, she loses immediately since Bob will take the rest. If she takes away 1, leaving $7 = 5 + 2$, then Bob takes away 2. If she takes away 2, leaving $6 = 5 + 1$, then Bob takes away 1. Allie loses if she takes 2, so she takes 1, leaving $4 = 3 + 1$. Bob takes away 1 and Allie will now lose if she removes either 2 or 1. Each time Bob partitions the remainder into non-adjacent Fibonacci numbers and takes the least of those. If n had been 15, Allie could take away 2, leaving 13 and the tables are reversed, and so she can now win by following his strategy. If n had been $32 = 21 + 8 + 3$, Allie can take away 3 and by taking away the smallest Fibonacci number in each partition, she will eventually become the second player in a game starting with a Fibonacci number and will win.

To establish Bob's winning strategy, we must establish 2 things: 1) that it is always possible to remove the smallest Fibonacci number in a partition of a remaining amount, and 2) if a player is able to remove the smallest Fibonacci number in a partition of a remaining amount, then that player has a winning strategy.

Lemma 1: If $s < F_i$, then the smallest Fibonacci number in the partition of $F_i - s$ is less than $2s$.

(This means that the smallest F_k can be chosen.)

Lemma 2: If a player is faced with q matches and is allowed to remove the smallest Fibonacci number in the partition of q, then that player has a winning strategy.

These two lemmas solve the problem in this way: if n is not a Fibonacci number, then Allie can remove the smallest Fibonacci number in the partition of n and force a win by Lemma 2. If n is a Fibonacci number F_n, then Allie must remove a number s which is less than F_n. By Lemma 1, Bob is able to remove the smallest Fibonacci number in the partition of $F_n - s$ and then he can force a win by Lemma 2.

Proof Lemma 1: Let $F_n - s = F_{a_1} + \ldots + F_{a_k}$ for non-adjacent decreasing F_{a_i}. We must show that

$s > \frac{1}{2}F_{a_k} \Leftrightarrow F_n - s + \frac{1}{2}F_{a_k} < F_n$ which would follow from $F_{a_1} + \ldots + F_{a_k} + \frac{1}{2} \cdot F_{a_k} < F_{a_1+1}$.

But by a short induction and the fact (*) that $F_n > \frac{1}{2} \cdot F_{n+1}$ we have:

$$
\begin{aligned}
F_{a_1+1} &= F_{a_1} + F_{a_1-2} + F_{a_1-4} + \ldots + F_{a_1-2(k-1)} + F_{a_1-2k+1} \\
&\geq F_{a_1} + F_{a_1-2} + F_{a_1-4} + \ldots + F_{a_1-2(k-1)} + \frac{1}{2}F_{a_1-2(k-1)} \text{ by (*)} \\
&\geq F_{a_1} + F_{a_2} + \ldots + F_{a_k} + \frac{1}{2} \cdot F_{a_k}.
\end{aligned}
$$

Proof Lemma 2: We prove this by induction on q. If $q = 1$ the result is clearly true. Assume the result for all positive integers less than q. Let $q = F_{a_1} + \ldots + F_{a_k}$ where F_{a_i} are non-adjacent. Then the first player removes F_{a_k} matches leaving $q - F_{a_k} = F_{a_1} + \ldots + F_{a_{k-1}}$. If $k = 1$, the first player has won. Otherwise, the next player can remove up to $2 \cdot F_{a_k}$ matches, a number that is less than F_{a_k-1} since $F_{a_k-1} \geq F_{a_k+2} = F_{a_k+1} + F_{a_k} > F_{a_k} + F_{a_k}$ for $a_k \geq 2$. So, the second player must move to $q - F_{a_k} - s = F_{a_1} + \ldots + F_{a_{k-2}} + T$ where T is a sum of non-adjacent Fibonacci numbers smaller than $F_{a_{k-1}}$. By Lemma 1, the first player can remove the smallest Fibonacci number in the partition of T which is the smallest Fibonacci number in the partition of $F_{a_1} + \ldots + F_{a_{k-2}} + T = q - F_{a_k} - s$ matches and so by the inductive hypothesis that player can force a win.

I–1. Numbers are to be placed in the nine squares so that the sum of the elements of each row and of each column are equal. If 1, 9, 9, and 8 are placed as shown, compute $X + Y - Z$.

1	Y	9
9	X	Z
		8

I–2. Coefficients a, b, and c of $ax^2 + bx + c = 0$ are selected without replacement from $N = \{-4, -3, -2, -1, 1, 2, 3, 4\}$. Compute the probability that $x = 1$ is a solution of the equation.

I–3. If the sum of the positive three-digit base ten numbers ABC, CAB, and BCA can be factored into four distinct primes, compute the largest possible value of the product $A \cdot B \cdot C$.

I–4. The sides of a non-right isosceles ΔABC are $\sin x$, $\cos x$, and $\tan x$. Compute $\sin x$.

I–5. Compute: $199919981997^2 - 2 \cdot 199919981994^2 + 199919981991^2$.

I–6. Let x_n be the remainder when x is divided by n. For positive integral x, compute the sum of all elements in the solution set of: $x^5(x_5)^5 - x^6 - (x_5)^6 + x(x_5) = 0$

I–7. Compute the smallest number d such that fewer than half of the positive integers with d-digits have all distinct digits.

I–8. In <u>concave</u> hexagon $ABCDEF$, $m\angle A = m\angle B = m\angle C = 90°$, $m\angle D = 100°$, and $m\angle F = 80°$. Also, $CD = FA$, $AB = 7$, $BC = 10$, and $EF + DE = 12$. Compute the area of the hexagon.

1. 15

2. $\dfrac{1}{14}$

3. 648

4. $\dfrac{-1+\sqrt{5}}{2}$

5. 18

6. 1300

7. 5

8. $\dfrac{145}{4} = 36.25$

I–1. The top row and the right hand column give $1 + Y + 9 = 9 + Z + 8 \rightarrow$ $Y - Z = 7$. Using the left hand column and the bottom row, $1 + 9 + A = A + B + 8 \rightarrow B = 2$. Thus $1 + Y + 9 = Y + X + 2 \rightarrow$ $X = 8$, making $X + Y - Z = \boxed{15}$.

1	Y	9
9	X	Z
A	B	8

I–2. If $x = 1$ is a solution then $a + b + c = 0$. There are ${}_8C_3 = 56$ unordered triples of a, b, and c, and of those, the following unordered triples satisfy $a + b + c = 0$: $(-4, 1, 3)$, $(4, -1, -3)$, $(-3, 1, 2)$ and $(3, -1, -2)$. Thus, the probability is $\dfrac{4}{56} = \boxed{\dfrac{1}{14}}$.

I–3. $(100A + 10B + C) + (100C + 10A + B) + (100B + 10C + A) = 111(A + B + C) = 3 \cdot 37(A + B + C)$. The maximum value of $A + B + C = 27$, but that is not factorable into the product of two primes. Let $A + B + C = 26 = 2 \cdot 13$, making the sum equal to $2 \cdot 3 \cdot 13 \cdot 37$. Let $A = 9$, $B = 9$, $C = 8$, making $A \cdot B \cdot C = 9 \cdot 9 \cdot 8 = \boxed{648}$.

I–4. If $\sin x = \cos x$, then $x = 45°$, the third side $= \tan 45° = 1$, but the sides are $\dfrac{1}{\sqrt{2}}$, $\dfrac{1}{\sqrt{2}}$, and 1, making a right triangle. If $\sin x = \tan x$, then $\sin x = 0$ or $\cos x = 1$ which makes $\sin x = 0$ and no triangle can be formed. Thus, $\cos x = \tan x \rightarrow \cos^2 x = \sin x \rightarrow 1 - \sin^2 x = \sin x \rightarrow \sin^2 x + \sin x - 1 = 0 \rightarrow$ $\sin x = \boxed{\dfrac{-1 + \sqrt{5}}{2}}$.

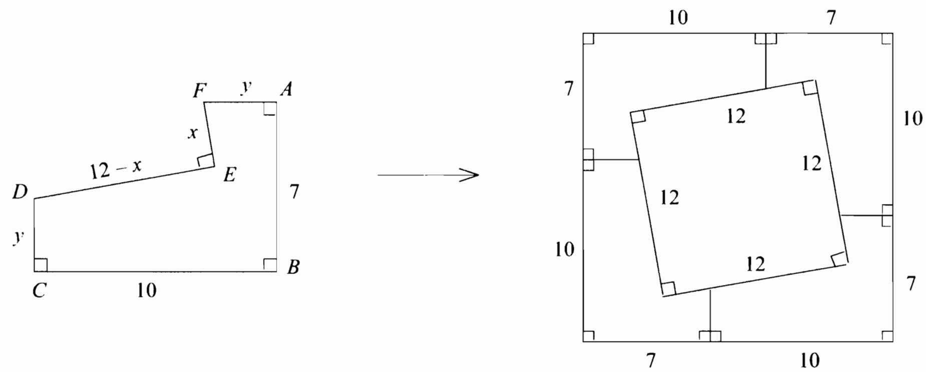
I–5. Let $p = 199919981994$. Then the expression can be written as $(p + 3)^2 - 2p^2 + (p - 3)^2 =$ $p^2 + 6p + 9 - 2p^2 + p^2 - 6p + 9 = \boxed{18}$.

I–6. Since $x^5(x_5)^5 - x^6 - (x_5)^6 + x \cdot (x_5) = \left((x_5)^5 - x\right)\left(x^5 - x_5\right) = 0$, then $(x_5)^5 = x$ or $x^5 = x_5$ where $x_5 = 0, 1, 2, 3,$ or 4. Hence, from the first equation $x = 1^5, 2^5, 3^5$ or 4^5 and from the second equation $x = 1$. The sum of the elements in the solution set is $1 + 32 + 243 + 1024 = \boxed{1300}$.

I–7. The table shows the fraction of d-digit numbers with distinct digits:

$d = 1$	$9/9 = 1$	
$d = 2$	$(9/9)\cdot(9/10) = 0.9$	
$d = 3$	$(9/9)\cdot(9/10)\cdot(8/10) = 0.72$	
$d = 4$	$(9/9)\cdot(9/10)\cdot(8/10)\cdot(7/10) = 0.504$	
$d = 5$	$(9/9)\cdot(9/10)\cdot(8/10)\cdot(7/10)\cdot(6/10) = 0.3024$	$\rightarrow$ The answer is $\boxed{5}$.

I–8. Method #1: Since the sum of the interior angles of a hexagon is 720°, then $360° - m\angle FED = 720° - 3\cdot 90° - 100° - 80° = 270°$, so $m\angle FED = 90°$. Duplicate the hexagon, rotate each copy through a multiple of 90° and obtain the figure shown below, a square of side 12 oriented inside a square of side 17. The area of the hexagon is $\frac{1}{4}\left(17^2 - 12^2\right) = \boxed{\frac{145}{4} = 36.25}$.



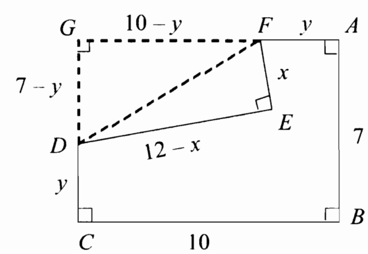
Method #2: Complete rectangle $ABCG$ as shown. Then

$a(ABCG) = a(ABCDEF) + a(\Delta DEF) + a(\Delta FGD)$ giving

$$70 = a(ABCDEF) + \frac{1}{2}\cdot x\cdot(12 - x) + \frac{1}{2}(7 - y)(10 - y)$$

$$140 = 2(a(ABCDEF)) + 12x - x^2 + 70 - 17y + y^2$$

$$70 = 2(a(ABCDEF)) + 12x - x^2 - 17y + y^2.$$



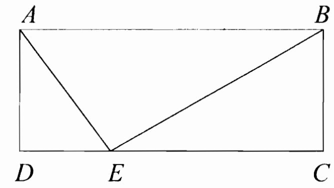
Also, since $\overline{FD}$ is the hypotenuse of both ΔFED and ΔFGD we have $x^2 + (12 - x)^2 = (10 - y)^2 + (7 - y)^2$ which yields $2x^2 - 24x + 144 = 149 - 34y + 2y^2$ which simplifies to $x^2 - 12x = \frac{5}{2} - 17x + y^2$. Replacing $12x - x^2$ in the first equation by $-\left(\frac{5}{2} - 17x + y^2\right)$ gives

$70 = 2(a(ABCDEF)) - \left(\frac{5}{2} - 17x + y^2\right) - 17x + y^2$. The x-terms and the y^2-terms cancel yielding

$70 + \frac{5}{2} = 2(a(ABCDEF)) \rightarrow$ the area of $ABCDEF$ equals $\frac{145}{4}$.

ARML Relay #1 – 1998

R1–1. In rectangle *ABCD*, the areas of ΔADE, ΔCBE, and ΔAEB form an arithmetic sequence. If the ratio of the area of the largest region to the smallest can be expressed in simplest terms as $\frac{m}{n}$, compute $m + n$.



R1–2. Let T = TNYWR. The digits T, $T+1$, $T+2$, and $T+3$ are placed at random in the indicated spaces: 1 __ 9 __ 9 __ 8 __ . Each digit is used, one per space. Compute the probability that the resulting number is divisible by 3 or 5.

R1–3. Let T = TNYWR. Compute $\frac{x}{y}$ in simplest terms if $x + \frac{1}{y} = \frac{1}{T}$ and $y + \frac{1}{x} = T$.

ARML Relay #2 – 1998

R2–1. A box 4 by 6 by 8 is resting on the floor. A box 2 by 3 by 5 is placed on top of the first box forming a two-box tower. If *A* is the exposed surface of the tower, compute the least possible value of the area of *A*.

R2–2. Let T = TNYWR and let $K = \frac{T-12}{4}$. In the regular *n*-gon $A_1A_2 \ldots A_n$, the measure of $\angle A_2A_1A_4 = K$.

Compute the value of *n*.

R2–3. Let T = TNYWR and let f be a linear function with positive slope passing through the origin. Compute the least value of $f(5T) + f^{-1}(5T)$.

Relay #1:

R1–1. 4

R1–2. $\frac{1}{4}$

R1–3. 16

Relay #2:

R2–1. 192

R2–2. 8

R2–3. 80

Solutions to ARML Relay #1 – 1998

R1–1. Let the area of $\Delta ADE = a$, the area of $\Delta CBE = a + d$, and the area of $\Delta AEB = a + 2d$. Since the area of ΔAEB is half the rectangle, the sum of the areas of the smaller two triangles equals the area of ΔAEB. Thus, $a + (a + d) = a + 2d \rightarrow a = d \rightarrow$ the areas are a, $2a$, and $3a$. So, $3a/a \rightarrow 3 + 1 = \boxed{4}$.

R1–2. $T = 4$. Since $1 + 9 + 9 + 8 = 27$, the number is divisible by 3 if $T + (T + 1) + (T + 2) + (T + 3) = 4T + 6$ is divisible by 3. The number is divisible by 5 if the last digit is 0 or 5. Without knowing T, the following probabilities can be calculated:

T	T+1	T+2	T+3		T	T+1	T+2	T+3	
0	1	2	3	$P = 1$	1	2	3	4	$P = 0$
2	3	4	5	$P = 1/4$	3	4	5	6	$P = 1$
4	5	6	7	$P = 1/4$	5	6	7	8	$P = 1/4$
6	7	8	9	$P = 1$					

Since $T = 4$, the probability equals $\boxed{\frac{1}{4}}$.

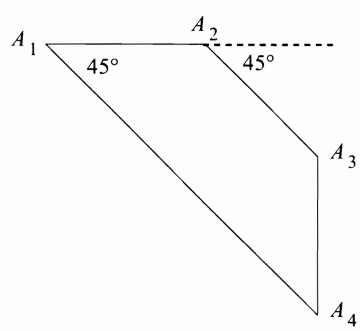
R1–3. $T = \frac{1}{4}$. Subtracting $xy + 1 = xT$ from $xy + 1 = \frac{Y}{T}$ yields $\frac{Y}{T} = xT$ giving $\frac{x}{y} = \frac{1}{T^2} = \boxed{16}$.

Solutions to ARML Relay #2 – 1998

R2–1. The minimum exposed area occurs if both boxes are placed with the side of largest area face down. Note that the 3 by 5 side of the small box is hidden and also hides an area of 3 by 5 on the top face of the larger box. The minimum area is:

$$2(4 \cdot 6 + 4 \cdot 8 + 6 \cdot 8) - 6 \cdot 8 + 2(2 \cdot 3 + 2 \cdot 5 + 3 \cdot 5) - 2(3 \cdot 5) = \boxed{192}.$$

R2–2. $T = 192 \rightarrow K = 45°$. Since $\overline{A_1A_4} \parallel \overline{A_2A_3}$, the measure of the exterior angle at A_2 equals $m\angle A_1$. Thus, $K = \frac{360}{n}$ making $n = \frac{360}{K} = \frac{360}{45} = \boxed{8}$.



R2–3. $T = 8$. Given $f: y = mx$, $f^{-1}: x = my \rightarrow y = \frac{x}{m}$. Hence, $f(5T) + f^{-1}(5T) = 5Tm + \frac{5T}{m} = 5T\left(m + \frac{1}{m}\right)$. Since the minimum of $m + \frac{1}{m} = 2$, the minimum of $f(5T) + f^{-1}(5T) = 10T = \boxed{80}$.

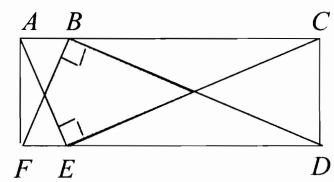
Note: Pass answers from position 1 to position 15.

1. At which value of x does $y = (x-1)(x-19)+7$ take on its minimum value?

2. Let T = TNYWR. If $a_n = \dfrac{a_{n-1}+3T}{2} - a_{n-2}$ and all a_i are equal, compute a_{1998}.

3. Let T = TNYWR. An auditorium has T rows of T seats and each row is numbered from 1 to T by 1's. Compute the number of even-numbered seats.

4. Let T = TNYWR. In rectangle *ACDF*, $AC = 4T$ and $CD = T$. Right triangle *FBD* is congruent to ΔAEC. Compute the ratio of the area of the shaded region to the area of *ACDF*.

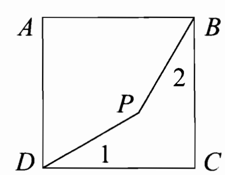


5. Let T = TNYWR and set $K = \dfrac{1}{T}$. If the equation $x = \dfrac{K}{K-x}$ has a unique solution, compute x.

6. Let T = TNYWR. The first term of an infinite geometric series is T and the limit of the sum is 10. Compute the common ratio.

7. Let T = TNYWR. Compute the number of factors of $\left(10^4\right)\cdot T$.

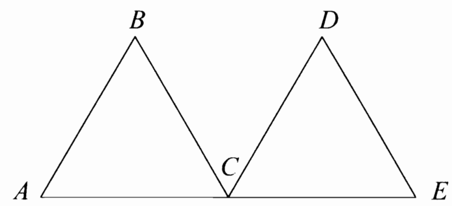
8. Let T = TNYWR and set $K = \dfrac{T}{7}$. In square *ABCD*, $DP = BP = 2K$, $m\angle 1 = m\angle 2 = 30°$. The area of the square can be written in simplest form as $a + b\sqrt{3}$. Compute $a + b$.



9. Let T = TNYWR, $T \neq 1$. If the ordered pair (x, y) is the solution to the system, compute $x + y$.

$$x + 2y = 3$$
$$Tx + (T+1)y = T+2$$

10. Let T = TNYWR. Shown are two equilateral triangles ABC and CDE of side T.
If $AD = \sqrt{N}$, compute N.

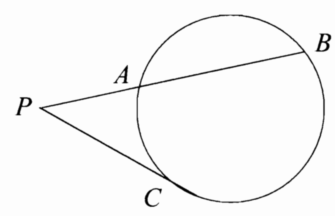


11. Let T = TNYWR. If $\sin\theta + \cos\theta = \frac{1}{T}$, compute $\sin 2\theta$.

12. Let T = TNYWR. Let T be the slope of a line passing through (–14, 16). Compute the x-intercept.

13. Let T = TNYWR. Compute the number of integers from 1 to 10^T inclusive that are divisible by 4 or 5.

14. Let T = TNYWR and set $AB = \frac{4T}{1000} - 1$. $\overline{PC}$ is tangent to the circle, $PA = \frac{x}{2}$, and $PC = x$. Compute x.



15. Let T = TNYWR. Compute the largest value of x satisfying $x^2 - Tx + y^2 - 8y = 8$.

1. 10
2. 10
3. 50
4. $\frac{1}{4}$
5. 2
6. $\frac{4}{5}$
7. 28
8. 96
9. 1
10. 3
11. $-\frac{8}{9}$
12. 4
13. 4000
14. 10
15. 12

1. Minimum occurs at $x = -\frac{b}{2a}$. Since $y = x^2 - 20x + 26$, the minimum occurs at $x = \boxed{10}$.

2. $a_n = \frac{a_n + 3T}{2} - a_n \to 2a_n = a_n + 3T - 2a_n \to 3a_n = 3T \to a_n = T$. $\therefore\ a_{1998} = \boxed{10}$

3. If T is even there are $\left(\frac{T}{2}\right) \cdot T$ even-numbered seats and if T is odd there are $\left(\frac{T-1}{2}\right) \cdot T$ even-numbered seats.

   $T = 10$ gives $5 \cdot 10 = \boxed{50}$ even-numbered seats.

4. T is irrelevant. Connect B and E. The area of the shaded region lying in $ABEF$ is $\frac{1}{4} \cdot a(ABEF)$. The area of the shaded region in $BEDC$ is $\frac{1}{4} \cdot a(BEDC)$ so the ratio of the shaded region to $ACDF$ is $\boxed{\frac{1}{4}}$.

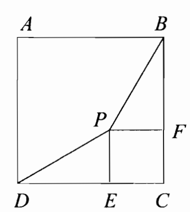
5. Simplifying, we obtain $Kx - x^2 = K \to x^2 - Kx + K = 0$. The discriminant is $K^2 - 4K$ which gives a unique value for x if $K = 0$ or 4. If $K = 0, x = 0$. If $K = 4, x = 2$. Since $T = \frac{1}{4}, K = 4$, making $x = \boxed{2}$.

6. Set $10 = \frac{T}{1-r}$, then $r = \frac{10 - T}{10}$. Since $T = 2, r = \boxed{\frac{4}{5}}$.

7. $T = \frac{4}{5}$ and $10{,}000 \cdot \frac{4}{5} = 8000 = 8 \cdot 1000 = 2^6 \cdot 5^3$. There are $7 \cdot 4 = \boxed{28}$ factors.

8. $K = 4$. Since $DP = BP = 2K$, then $PE = K$, $DE = K\sqrt{3}$ and side $DC = K\sqrt{3} + K$. Then $DC^2 = \left(K\sqrt{3} + K\right)^2 = 4K^2 + 2K^2\sqrt{3}$.

   So, $a + b = 6K^2 = 6 \cdot 4^2 = \boxed{96}$.



9. Multiply the top equation by T and subtract giving $2Ty - (Ty + y) = 2T - 2$ which gives $y(T - 1) = 2(T - 1)$ so $y = 2$. Substitution yields $x = -1$, so the value of T is irrelevant and $x + y = \boxed{1}$.

10. Drop an altitude from D to point F on $\overline{CE}$. Then $AF = \dfrac{3T}{2}$ and $BF = \dfrac{T\sqrt{3}}{2}$. Thus,

$AD^2 = \left(\dfrac{3T}{2}\right)^2 + \left(\dfrac{T\sqrt{3}}{2}\right)^2 = 3T^2$. Since $T = 1$ and $N = AD^2$, then $N = \boxed{3}$.

11. $(\sin\theta + \cos\theta)^2 = \dfrac{1}{T^2} \to \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1 + \sin 2\theta = \dfrac{1}{T^2} \to \sin 2\theta = \dfrac{1}{T^2} - 1$.

Since $T = 3$, $\sin 2\theta = \dfrac{1}{9} - 1 = \boxed{-\dfrac{8}{9}}$.

12. $y - 16 = T(x + 14)$. Let $y = 0$. Then $x = -\dfrac{16}{T} - 14$. Since $T = -\dfrac{8}{9}$, $x = -16\left(-\dfrac{9}{8}\right) - 14 = \boxed{4}$.

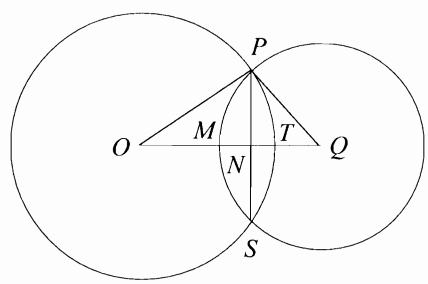
13. If $T = 1$, then there are 4 divisors. If $T = 2$, then $\dfrac{100}{4} + \dfrac{100}{5} - \dfrac{100}{20} = 40$ divisors. If $T = 3$, then $\dfrac{1000}{4} + \dfrac{1000}{5} - \dfrac{1000}{20} = 400$ divisors, if $T = 4$, then $\dfrac{10000}{4} + \dfrac{10000}{5} - \dfrac{10000}{20} = 4000$ divisors, and so on. Here $T = 4$ so the answer is $\boxed{4000}$.

14. $AB = 15$. Since $PA \cdot PB = PA(PA + AB) = PC^2$, then $\left(\dfrac{x}{2}\right) \cdot \left(\dfrac{x}{2} + 15\right) = x^2 \to \dfrac{x}{2} + 15 = 2x$. Thus,

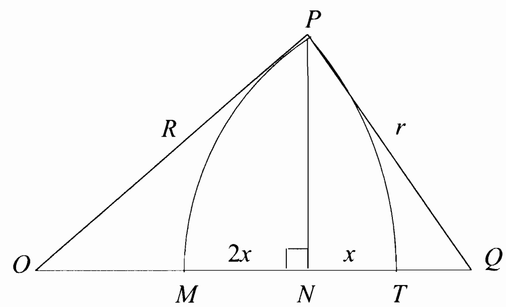
$15 = \dfrac{3x}{2} \to x = \boxed{10}$.

15. $x^2 - Tx + \dfrac{T^2}{4} + y^2 - 8y + 16 = 24 + \dfrac{T^2}{4} \to \left(x - \dfrac{T}{2}\right)^2 + (y - 4)^2 = \dfrac{T^2 + 96}{4}$. The largest value for x occurs when the expression involving y is 0, so let $y = 4$, then $x = \dfrac{T + \sqrt{T^2 + 96}}{2}$. Nice values result if $T = 2, 5, 10, 23, \ldots$. Since $T = 10$, then $x = \boxed{12}$.

1. Let circles O and Q have a common chord $\overline{PS}$. If $OQ = 324$ and $MN : NT = 2 : 1$, compute $OP - PQ$.



1. Let the radius of circle O be R and the radius of circle Q be r. Let $MN = 2x$ and $NT = x$. Then $ON = R - x$ and $QN = r - 2x$. Using ΔONP we obtain $(R-x)^2 + h^2 = R^2$ and using ΔQNP we obtain $(r-2x)^2 + h^2 = r^2$. The two equations simplify to $x^2 - 2Rx + h^2 = 0$ (1) and $4x^2 - 4rx + h^2 = 0$ (2)



Subtracting (1) from (2) and dividing by x yields the condition: $3x = 4r - 2R$.

Since $OQ = 324$, then $OQ = R + r - 3x = 324 \rightarrow 3x = R + r - 324$.

Thus, $R + r - 324 = 4r - 2R \rightarrow 3R - 3r = 324 \rightarrow R - r = \boxed{108}$.

ARML

1999

One of the most exciting developments this year was the establishment of a contest modeled on ARML Taiwan. In cooperation with the Nine Nine Cultural and Educational Foundation, ARML helped establish the Taiwan Regional Mathematics League called TRML. The first contest was held in August, 1999. This year there were 24 teams in Division A and 81 teams in Division B. From these 105 teams some 1575 students participated. For the national title, San Francisco Bay A narrowly edged out Massachusetts A. The difference was the score on the individual round. We had three teams from Taiwan competing this year and the C team earned the top score in Division B.

John Goebel of North Carolina received the Samuel Greitzer Distinguished Coach Award. John has played an active role in support North Carolina's ARML team and also has been crucial to the development and success of the North Carolina Mathematics League.

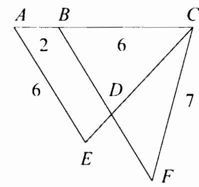
Tom Kilkelly of Minnesota and Richard Kalman of New York City received the Alfred Kalfus Founder's Award.

Tom has helped immeasurably with the Minnesota ARML team and has helped create and sustain the ARML Power Contest. He has written most of the contests, he has done much of the grading, and, of course, he has been responsible for organizing the contest.

Richard published the *ARMLog* for 10 years and since the mid 1980's he has chaired the Power Question grading at Penn State. *ARMLog* provided information about the contest and its results and also introduced readers to the coaches and organizers of the meet as well as providing articles on mathematics and mathematics education. Richard did a wonderful job of pulling this all together.

Kurt Lazaro of Maine received the Zachary Sobol award for his contributions to his team.

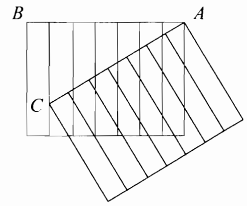
T–1. If $AB = 2, BC = 6, AE = 6, BF = 8, CE = 7$, and $CF = 7$, compute the ratio of the area of quadrilateral $ABDE$ to the area of ΔCDF.



T–2. Compute the number of ordered triples of integers (x, y, z), $1729 < x, y, z < 1999$ which satisfy:

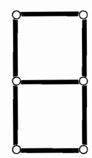
$$x^2 + xy + y^2 = y^3 - x^3 \quad \text{and} \quad yz + 1 = y^2 + z.$$

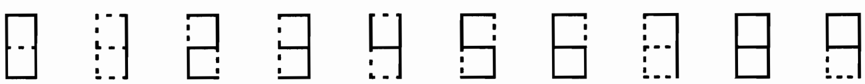
T–3. Two identical sheets of paper with 8 equally spaced lines are attached at corner A. Initially, corners B and C coincide. The top piece is then rotated until corner C lies on the next–to–last line of the bottom sheet. If $BC = \sqrt{28}$, compute AB.



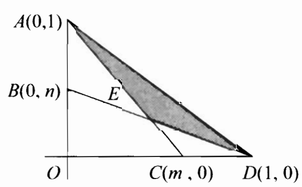
T–4. Define $\lfloor n \rfloor$ to be the greatest integer less than or equal to n. Compute the area of the solution set of $\lfloor x \rfloor \cdot \lfloor y \rfloor = 2000$.

T–5. A digital watch displays the digits from 0 to 9 as shown below by displaying some subset of the seven segments which make up an 8 as shown at the right. If a randomly chosen segment fails to light up, compute the expected value of the number of digits that can still be displayed.





T–6. Let the area of $\Delta AED = R$ and the area of $OCEB = S$. For $0 < m, n < 1$, express $|R - S|$ in terms of m and n.



T–7. Define a sequence of integers as follows: $a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 5, a_5 = 7, a_6 = 9$, the next four terms are the next four even integers after 9, the next five terms are the next five odd integers after 16, and so on. Compute a_{1999}.

T–8. There was a chess tournament involving two eighth graders and at least ten ninth graders. Each contestant played once against each other contestant. Each contestant received two points for a win, one point for a tie, and zero points for a loss. The two eighth graders amassed a total of 20 points and each ninth grader earned N points. Compute N.

T–9. Compute the number of distinct ways one can arrange the numbers 21, 31, 41, 51, 61, 71, and 81 from left to right so that the sum of every four consecutive numbers is divisible by 3.

T–10. Let $N_b = 1_b + 2_b + \ldots + 100_b$ where b is an integer greater than 2. Compute the number of values of b for which the sum of the squares of the digits of N_b is at most 512.

1. 1 or 1:1
2. 267
3. $7\sqrt{2}$
4. 40
5. $\frac{22}{7}$
6. $\frac{1}{2}\left|1 - m - n\right|$
7. 3935
8. 20
9. 144
10. 30

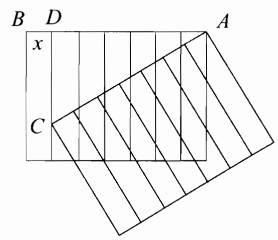
T–1. Since $AC = BF = 8$, $EC = CF = 7$, and $AE = BC = 6$, then $\Delta AEC \cong \Delta BCF$ by SSS. Subtracting the area of ΔBDC from each of AEC and BCF gives area of $ABDE$ = area of ΔCDF. Thus the ratio is $\boxed{1 \text{ or } 1:1}$.

T–2. Since $(y-x)(x^2+xy+y^2) = x^2+xy+y^2$, then either $y - x = 1$ or $x^2+xy+y^2 = 0$, but the latter is an impossibility for positive x and y. Thus, $y = x+1$. From the second equation, $yz - z = y^2 - 1$ gives $z(y-1) = (y-1)(y+1) \rightarrow y = 1$ (an impossibility), or $z = y+1$. Thus, $y = x+1$ and $z = x+2$, and we obtain $1729 < x < x+1 < x+2 < 1999$, making $1729 < x < 1997 \rightarrow 1730 \le x \le 1996$. Thus, there are $1996 - 1730 + 1 = 267$ triples of the form (x, y, z). Answer: $\boxed{267}$.

T–3. <u>Method 1</u>: Let $AB = 7x \rightarrow AD = 6x$. Then by the Law of Cosines,

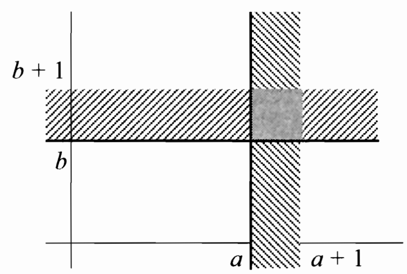
$BC^2 = 28 = 49x^2 + 49x^2 - 2(7x)(7x)\cos\angle BAC$. Since $\cos\angle BAC = \dfrac{6}{7}$,

we have $28 = 14x^2 \rightarrow x = \sqrt{2} \rightarrow AB = \boxed{7\sqrt{2}}$.



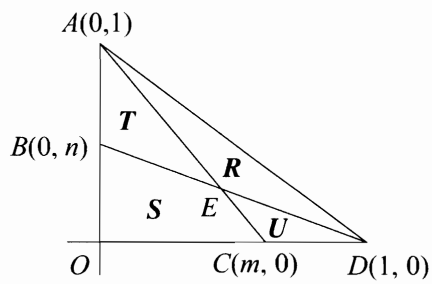
<u>Method 2</u>: If $AB = 7x$, then $AD = 6x$ and $AC = 7x$. By the Pythagorean Theorem on ΔADC, we obtain $DC = x\sqrt{13}$. By the Pythagorean Theorem on ΔBDC, we obtain $x = \sqrt{2} \rightarrow AB = 7\sqrt{2}$.

T–4. Clearly $[x]$ and $[y]$ will be integer factors of 2000. The graph of $[x] = a$ will be an infinitely long horizontal strip one unit wide, edgeless above. The graph of $[y] = b$ will be an infinitely long vertical strip one unit wide, edgeless to the right. The graph of the intersection of each pair of strips will be a unit square of area 1 with two edges missing. Thus, each factor pair generates a region of area 1. Since $2000 = 2^4 \cdot 5^3$, there are $5 \cdot 4 = 20$ positive factor pairs. Combined with 20 negative factor pairs, the area of the solution set equals $\boxed{40}$.



T–5. For a displayed digit d, let $u(d)$ represent the number of segments that can fail without impairing the display of the digit. Then $u(0) = 1$, $u(1) = 5$, $u(2) = 2$, $u(3) = 2$, $u(4) = 3$, $u(5) = 2$, $u(6) = 1$, $u(7) = 4$, $u(8) = 0$, and $u(9) = 2$. The sum of the number of pairs (segment, digit) such that the segment can be removed from the digit without impairing the display equals the sum of the above values, namely 1 + 5 + 2 + 2 + 3 + 2 + 1 + 4 + 0 + 2 = 22. There are 7 segments so the average number of displayable digits is $\boxed{\frac{22}{7}}$.

T–6. Label the regions as shown. Then:



1. $R + T = \text{area } \Delta ABD = \frac{1}{2} \cdot 1 \cdot (1 - n)$.

2. $R + U = \text{area } \Delta ACD = \frac{1}{2} \cdot 1 \cdot (1 - m)$.

3. $R + S + T + U = \text{area } \Delta AOD = \frac{1}{2}$.

Since $(R + T) + (R + U) - (R + S + T + U) = R - S = \frac{1}{2}(1 - n + 1 - m - 1)$, then $|R - S| = \boxed{\frac{1}{2}|1 - m - n|}$.

T–7. Method 1: By listing the terms out in staircase fashion, it is clear that the nth row ends in n^2 and that this entry is the $\frac{n(n+1)}{2}$ term of the sequence. For example, the 3[rd] row ends in 9 and this entry is the $\frac{3(4)}{2} = 6$th term in the sequence. We'll call it a_6. If $n = 62$, then $\frac{62(63)}{2} = 1953$ while if $n = 63$, then $\frac{63(64)}{2} = 2016$. Thus the 62nd row ends in $a_{1953} = 62^2 = 3844$. As a result, the 63rd row starts with $a_{1954} = 3845$ and this gives $a_{1999} = 3845 + 2(1999 - 1954) = \boxed{3935}$.

$1 = a_1$
$2 \quad 4 = a_3$
$5 \quad 7 \quad 9 = a_6$
$10 \quad 12 \quad 14 \quad 16 = a_{10}$
$17 \quad 19 \quad 21 \quad 23 \quad 25 = a_{15}$

Method 2: The sequence could be defined recursively: $a_1 = 1$ and for $n \geq 1$, $a_{n+1} = 1 + a_n$ if a_n is a triangular number or $2 + a_n$ otherwise. This simply says that the terms increase by 2 within a row and increase by 1 between rows. To get from a_1 to a_{1999} requires 1998 additions of either 1 or 2. Since a_{1999} lies in the 63rd row, it was obtained by 62 additions of 1 and 1998 – 62 = 1936 additions of 2. Thus, $a_{1999} = 1 + 62 + 1936 \cdot 2 = 3935$.

T–8. Let x be the number of 9th graders and let N be the number of points each 9th grader scored. Then there are $x + 2$ competitors and the total number of points is $Nx + 20$. Since each person plays each other person once, there are $\frac{(x+2)(x+1)}{2}$ games played and at two points per game there are $(x+2)(x+1)$ total points possible. Thus, we wish to find solutions to $Nx + 20 = (x+2)(x+1)$ for $x \geq 10$. Write the equation as $18 = x^2 + 3x - Nx = x(x + 3 - N)$. Since x and $(x + 3 - N)$ must be factors of 18 and $x \geq 10$, then $x = 18$, making $1 = 18 + 3 - N \to N = \boxed{20}$.

T–9. We need only consider the problem under (mod 3): we rewrite 21, 31, 41, 51, 61, 71, and 81 as 0, 1, 2, 0, 1, 2, 0. Suppose that $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ is a required arrangement. Then, $a_1 + a_2 + a_3 + a_4 \equiv a_2 + a_3 + a_4 + a_5 \equiv a_3 + a_4 + a_5 + a_6 \equiv a_4 + a_5 + a_6 + a_7 \equiv 0 \pmod 3$. Hence, $a_1 \equiv a_5$, $a_2 \equiv a_6$, and $a_3 \equiv a_7 \pmod 3$. Since there are only two numbers equal to 1(mod 3) or 2(mod 3), then a_4 must equal 0(mod 3). Thus, there are 3 choices for a_4. There are $3 \cdot 2$ choices for a_1, 2 choices from each of 0(mod 3), 1(mod 3), or 2(mod 3). There are $2 \cdot 2$ choices for a_2, $1 \cdot 2$ choices for a_3, and the rest are determined. Answer: $(6)(4)(2)(3) = \boxed{144}$.

T–10. Since $100_b = 1 \cdot b^2 + 0 \cdot b + 0 = b^2$, the sum of $1_b + 2_b + \ldots + 100_b$ can be expressed in base 10 as $1 + 2 + \ldots + b^2 = \frac{b^2\left(b^2+1\right)}{2} = \frac{b^4}{2} + \frac{b^2}{2}$. Now the trick is to rewrite in terms of base b. If b is even, we have $\left(\frac{b}{2}\right)b^3 + \left(\frac{b}{2}\right)b$ which gives a number whose digits in order are $\boxed{\frac{b}{2}}\boxed{0}\boxed{\frac{b}{2}}\boxed{0}$. Thus, we want $\left(\frac{b}{2}\right)^2 + \left(\frac{b}{2}\right)^2 \leq 512 \to b^2 \leq 1024 \to 4 \leq b \leq 32$ for b even, giving 15 values for b. If b is odd, rewrite the expression as $\left(\frac{b-1}{2}\right)b^3 + \left(\frac{b+1}{2}\right)b^2$ giving a number whose digits are $\boxed{\frac{b-1}{2}}\boxed{\frac{b+1}{2}}\boxed{0}\boxed{0}$. Thus, we want $\left(\frac{b-1}{2}\right)^2 + \left(\frac{b+1}{2}\right)^2 \leq 512 \to b^2 + 1 \leq 1024$, making b an odd number such that $3 \leq b \leq 31$, yielding 15 values for b. Answer: $\boxed{30}$.

Suppose you are watching a figure skating contest in which the judges rank the contestants in order. On the left is your own ranking of the contestants and on the right is the judges' ranking:

Skater:	1	2	3	4
You:	1	2	3	4

Skater:	1	2	3	4
Judge 1:	1	2	4	3
Judge 2:	3	4	1	2
Judge 3:	2	1	3	4

For each pair of contestants, your relative ranking agrees with that of exactly two of the three judges. For example, you rate skater 3 better than 4, which agrees with judge 2 and judge 3, but not judge 1. To make the final rating, the sum of the three judges' rankings are used:

Skater:	1	2	3	4
Sum:	6	7	8	9

This ranking agrees with yours! This Power Question is devoted to determining why this is the case.

I) Turnabout:

Let $S = \{1, 2, 3, \ldots, n\}$. A permutation of S is a 1 to 1 function mapping the elements of S onto the elements of S. For example, if $S = \{1, 2, 3, 4\}$, then there is a permutation p of S such that $p(1) = 2$, $p(2) = 4$, $p(3) = 1$, and $p(4) = 3$. For the purposes of this Power Question we will write permutations by an ordered list of their values. Thus, for p above, we write p = 2 4 1 3, or simply, 2 4 1 3.

We define pair to mean two-element ordered tuples (j, k) so that $j < k$. Note that every instance of the word pair below refers to this definition and to no other definition. So for the purposes of this Power Question (3, 5) is a pair but (5, 3) is not a pair.

Given a permutation p of S, we define the turnabout set $T(p)$ of S to be the set of pairs (j, k) of elements of S so that $p(j) > p(k)$. We define the turnabout number $N(p)$ to be the number of pairs in $T(p)$.

Suppose p = 3 2 5 4 1. The turnabout set $T(p) = \{(1,2),(1,5),(2,5),(3,4),(3,5),(4,5)\}$, and the turnabout number of p is $N(p) = 6$. The turnabout set of a permutation represents those pairs of elements that are out of order and the turnabout number represents the "out-of-orderness" of a permutation.

1. Find the turnabout set T and the turnabout number N of the following permutations:

   a) 1 2 3 4 5 6 b) 3 1 4 5 9 2 6 8 7 c) 1 2 3 4 5 . . . n $(2n)$ $(2n-1)$ $(2n-2)$ . . . $(n+1)$

2. Find an example of:

   a) A permutation of 1 through 5 whose turnabout number is 1.

   b) A permutation of 1 through 6 whose turnabout number is 14.

   c) A permutation of 1 through 8 whose turnabout number is 14.

   d) A permutation of 1 through 10 whose turnabout number is 35.

3. For each of (a) through (c), give an example or prove that no example is possible.

   a) Find a permutation of 1 through 6 such that the turnabout set T consists of all pairs of the form (o, e) where o represents an odd number and e an even number.

   b) Find a permutation of 1 through 99 so that the turnabout set T consists of every pair of integers of the form $(10a + b, 10a + c)$, for $0 \le a, b, c \le 9$ with not both a and b equal to 0 and $b < c$.

   c) Find a permutation of 1 through 2000 so that the turnabout set T consists of all pairs with the same parity, i.e., both numbers are odd or both are even.

4. If T is a turnabout set of a permutation of $\{1, \ldots, n\}$, let $\bar{T}$ be the set of pairs that can be formed from $\{1, 2, \ldots, n\}$ which are not in T. Is $\bar{T}$ a turnabout set of some permutation? Prove your answer.

5. Show that two different permutations of $\{1, \ldots, n\}$ cannot have the same turnabout set T.

6. Show that for any integer t such that $0 \le t \le \dfrac{n(n-1)}{2}$, there exists a permutation of $\{1, \ldots, n\}$ whose turnabout number is t.

7. If T is the turnabout set of a permutation p, then it can be shown that $p(k)$ equals k plus the number of pairs (k, j) in T minus the number of pairs (j, k) in T.

   a) Verify this fact directly for the permutation 3 1 4 5 9 2 6 8 7 by finding T explicitly and calculating the right-hand side of the above equality for every k between 1 and 9.

   b) Prove this fact.

II) Fair Play:

Given a set of permutations p_1 through p_n define the functional sum of permutations $P = p_1 + \ldots + p_n$ to be the function $P(x) = p_1(x) + \ldots + p_n(x)$. So if $p_1 = 5\ 4\ 3\ 1\ 2$, $p_2 = 4\ 2\ 3\ 1\ 5$, and $p_3 = 2\ 3\ 1\ 4\ 5$, then $P = p_1 + p_2 + p_3 = 11\ 9\ 7\ 6\ 12$. Let $S = \{1, 2, \ldots, n\}$ be the domain of a functional sum P of n entries. In the example above, $P(1) = 11, P(2) = 9, P(3) = 7, P(4) = 6$, and $P(5) = 12$. We define the <u>turnabout set of P</u>, denoted $T(P)$, to be the set of pairs (j, k) of the elements of S so that $P(j) > P(k)$. We define the <u>turnabout number of P</u>, denoted $N(P)$, to be the number of pairs in $T(P)$.

8. a) Find three permutations of $\{1, \ldots, 5\}$ whose functional sum is 9 9 9 9 9.

   b) Find three permutations p_1, p_2, p_3 of $\{1, \ldots, 4\}$ so that no pair appears in more than one turnabout set of p_m, but so that the sum $P = p_1 + p_2 + p_3$ has a turnabout number N greater than 0. This shows that the condition that you disagree with exactly one of the judges is necessary.

9. Prove that if p_1 through p_3 are permutations of $\{1, \ldots, 4\}$, so that every pair appears exactly once in a turnabout set of p_m for some m, then the turnabout set of $P = p_1 + p_2 + p_3$ is empty. (Hint: use #7.) This corresponds to the original skater question.

10. Generalize problem 9 with proof. Maximum points for the best accurate generalization.

1a. Since $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 4$, $p(5) = 5$, and $p(6) = 6$, there are no ordered pairs (j, k) for which $p(j) > p(k)$, so $T = \varnothing$ and $N = 0$.

1b. Given:

1	2	3	4	5	6	7	8	9
3	1	4	5	9	2	6	8	7

$T = \{(1, 2), (1, 6), (3, 6), (4, 6), (5, 6), (5, 7), (5, 8), (5, 9), (8, 9)\}$ and $N = 9$.

1c.
$$\begin{aligned} T = \{&(n+1, n+2), & (n+1, n+3), & (n+1, n+4), & \ldots & (n+1, 2n),\\ & & (n+2, n+3), & (n+2, n+4), & \ldots & (n+2, 2n),\\ & & & (n+3, n+4), & \ldots & (n+3, 2n),\\ & & & & \ddots & \vdots\\ & & & & & (2n-1, 2n)\} \end{aligned}$$

There are $n - 1$ elements in the first row, $n - 2$ in the second, and so on down to 1 in the final row.

Thus, $N = 1 + 2 + \ldots + n - 1 = \dfrac{(1 - n - 1)(n - 1)}{2} = \dfrac{n(n-1)}{2}$.

2a. Reverse any two consecutive elements as in 1 2 3 5 4, or 2 1 3 4 5, or 1 3 2 4 5. All have $N = 1$.

2b. 6 5 4 3 1 2 yields $\{(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6)\}$.

2c. There is more than one correct answer. Here are two: 6 5 4 3 1 2 7 8 or 1 2 8 7 6 5 3 4.

2d. There is more than one correct answer. Here are two: 9 8 7 6 5 4 3 1 2 10 or 10 9 8 7 6 1 2 3 4 5.

3a. 4 1 5 2 6 3 yields $T = \{(1, 2), (1, 4), (1, 6), (3, 4), (3, 6), (5, 6)\}$.

3b. 9 8 7 6 5 4 3 2 1 19 18 17 . . . 11 10 29 28 . . . 21 20 39 38 . . . 31 30 . . . 99 98 97 . . . 91 90.

3c. No such permutation exists. Let $p = a\,b\,c\,d \ldots$ To have the pair (1, 3) we must have $a > c$. But we can't have either (1, 2) or (2, 3) so $a < b$ and $b < c$, making $a < c$, a contradiction.

4. 1st solution: Consider $p = 2\ 5\ 1\ 3\ 4$. $T = \{(1,3),(2,3),(2,4),(2,5)\}$, making $\overline{T} = \{(1,2),(1,4),$ $(1,5),(3,4),(3,5),(4,5)\}$. We discover that $q = 4\ 1\ 5\ 3\ 2$ generates $\overline{T}$, suggesting that interchanging each pair of numbers whose sum is 6 generates $\overline{T}$. Thus, if p is a permutation defined on $\{1, 2, \ldots, n\}$ with turnabout set T, define permutation q by $p(x) + q(x) = n+1 \rightarrow q(x) = n+1-p(x)$. We'll show that the turnabout set R of q is the complement of p, namely $\overline{T}$. If $(j, k) \in T$, then $j < k$ and $p(j) > p(k)$ giving $-p(j) < -p(k) \rightarrow n+1-p(j) < n+1-p(k) \rightarrow q(j) < q(k) \rightarrow (j,k) \notin R$. If $(j,k) \notin T$, then $j < k$ and $p(j) < p(k) \rightarrow -p(j) > -p(k) \rightarrow n+1-p(j) > n+1-p(k) \rightarrow$ $q(j) > q(k)$, putting (j, k) in the turnabout set R of q. Thus, $R = \overline{T}$. For completeness, one could verify that q is 1 to 1 and onto. It is 1 to 1 since if $q(j) = q(k)$, then $n+1-p(j) = n+1-p(k)$, giving $p(j) = p(k)$, and since p is 1 to 1, $j = k$, making q a 1 to 1 mapping. It is onto because for any m in $\{1, 2, \ldots, n\}$, pick a value of k so that $p(k) = n+1-m$. Then, $q(k) = n+1-p(k) = m$.

2nd solution: We can reverse the relative order of pairs of numbers by taking their reciprocals. Consider $p = 2\ 1\ 4\ 3$. Write the reciprocals $\frac{1}{2}\ \ 1\ \ \frac{1}{4}\ \ \frac{1}{3}$ and pair 1 with the smallest, i.e., $\frac{1}{4}$, 2 with next smallest and so on, obtaining $q = 3\ 4\ 1\ 2$. Since $T(p) = \{(1,2),(3,4)\}$ and $T(q) = \{(1,3),(1,4),(2,3),(2,4)\}$, $T(q)$ is clearly the complement of $T(p)$. Let $p = p(1)\ \ p(2)\ \ \ldots\ \ p(n)$. If T is the turnabout set of p, then $(j, k) \in T$ if for $j < k$, $p(j) > p(k)$. Consider the sequence of the reciprocals of p: $\frac{1}{p(1)}\ \ \frac{1}{p(2)}\ \ \ldots\ \ \frac{1}{p(n)}$. Pair 1 with the smallest, 2 with the next and so on, generating a permutation q of $\{1, 2, \ldots n\}$. If for $j < k$, $p(j) < p(k)$, then $\frac{1}{p(j)} > \frac{1}{p(k)}$, making $q(j) > q(k)$. Thus, (j, k) lies in the turnabout set of q but not in the turnabout set of p, and we have constructed a permutation of $\{1, 2, \ldots, n\}$ whose turnabout set is the complement of $T(p)$.

5. Suppose $T(p) = T(q)$ for permutations p and q. We will show that $p = q$. Let $m = p(k)$. Then there $m-1$ numbers ℓ so that $p(\ell) < p(k)$ which means $m-1$ numbers ℓ so that either (k, ℓ) is in $T(p)$ or (ℓ, k) is not in $T(p)$. This defines a relative order for pairs of p. Since $T(p) = T(q)$, the same statement holds for q and that means $m = q(k)$. Thus, for each element k of q there is the same relative ordering of pairs as p. Thus, $q = p$.

6. By induction:

1) For $\{1\}, p = 1$ is the only permutation, making $T = \varnothing, N = 0$, and clearly, $\frac{1 \cdot (1-1)}{2} = 0$.

2) Assume that for $\{1, 2, \ldots, n-1\}$ there are permutations whose turnabout sets contain pairs yielding turnabout numbers lying between 0 and $\frac{(n-1)(n-2)}{2}$ inclusively. Given any of those permutations, say $p = a_1 \; a_2 \; . \; . \; . \; a_{n-1}$, place n to the far right, producing $p = a_1 \; a_2 \; . \; . \; . \; a_{n-1} \; n$. Since n is the largest element, the turnabout number of this permutation is unchanged, making it a number between 0 and $\frac{(n-1)(n-2)}{2}$ inclusively. The permutation with the largest turnabout set is $p = n-1 \;\; n-2 \; . \; . \; . \; 2 \;\; 1$. Place n on the far right of the permutation giving $n-1 \;\; n-2 \; \ldots \; 2 \;\; 1 \;\; n$. This doesn't introduce any new pairs into the turnabout set. Now move n back, one place at time, producing permutations whose turnabout numbers increase by 1 each time:

$$N(n-1 \quad n-2 \ldots 2 \quad 1 \quad n) = \frac{(n-1)(n-2)}{2}$$

$$N(n-1 \quad n-2 \ldots 2 \quad n \quad 1) = \frac{(n-1)(n-2)}{2} + 1$$

$$N(n-1 \quad n-2 \ldots n \quad 2 \quad 1) = \frac{(n-1)(n-2)}{2} + 2$$

.

.

$$N(n \quad n-1 \quad n-2 \ldots 2 \quad 1) = \frac{(n-1)(n-2)}{2} + (n-1) = \frac{n(n-1)}{2}$$

Thus, for $\{1, 2, \ldots, n\}$ we have shown that the turnabout number t satisfies $0 \le t \le \frac{n(n-1)}{2}$.

7a. Given:

1	2	3	4	5	6	7	8	9
3	1	4	5	9	2	6	8	7

we have $T = \{(1,2), (1,6), (3,6), (4,6), (5,6), (5,7), (5,8), (5,9), (8,9)\}$. Thus, $p(1) = 3$ which equals $1 + 2 - 0$ since there are two pairs starting with 1 and none ending with 1. Similarly: $p(2) = 2 + 0 - 1 = 1, p(3) = 3 + 1 - 0 = 4, p(4) = 4 + 1 - 0 = 5, p(5) = 5 + 4 - 0 = 9$, $p(6) = 6 + 0 - 6 = 2, p(7) = 7 + 0 - 1 = 6, p(8) = 8 + 1 - 1 = 8$, and $p(9) = 9 + 0 - 2 = 7$.

7b. If $p(k) = m$, that means there are $m - 1$ values ℓ so that $p(k) > p(\ell)$. Of these, suppose t of these values ℓ are less than k and $m - 1 - t$ are greater than k. Then there are $k - 1 - t$ pairs of (ℓ, k) in the turnabout set since there are $k - 1$ pairs (k, ℓ) and t of them are not in the turnabout set. There are $m - 1 - t$ pairs (k, ℓ) in the turnabout set corresponding to those ℓ greater than k so that $p(k) > p(\ell)$. The desired sum is then $k + (m - 1 - t) - (k - 1 - t) = m$.

8a. One answer among many: 1 2 3 4 5 + 3 4 5 1 2 + 5 3 1 4 2 = 9 9 9 9 9.

8b. Shown is one answer:

p_1	1	2	3	4	$T = \varnothing$
p_2	1	2	3	4	$T = \varnothing$
p_3	4	1	2	3	$T = \{(1, 2), (1, 3), (1, 4)\}$
P	6	5	8	11	$T = \{(1, 2)\}$ and $N = 1$.

9. By problem 7,

$$\begin{aligned} P(k) &= p_1(k) + p_2(k) + p_3(k) \\ &= k + \#\text{ pairs } (k, j) \text{ in } T(p_1) - \#\text{ pairs } (j, k) \text{ in } T(p_1) \\ &\quad + k + \#\text{ pairs } (k, j) \text{ in } T(p_2) - \#\text{ pairs } (j, k) \text{ in } T(p_2) \\ &\quad + k + \#\text{ pairs } (k, j) \text{ in } T(p_3) - \#\text{ pairs } (j, k) \text{ in } T(p_3) \\ &= 3k - \#\text{ pairs } (j, k) \text{ in } T(p_1) \\ &\quad - \#\text{ pairs } (j, k) \text{ in } T(p_2) \\ &\quad - \#\text{ pairs } (j, k) \text{ in } T(p_3) \\ &\quad + \#\text{ pairs } (k, j) \text{ in } T(p_1) \\ &\quad + \#\text{ pairs } (k, j) \text{ in } T(p_2) \\ &\quad + \#\text{ pairs } (k, j) \text{ in } T(p_3) \end{aligned}$$

But every pair (ℓ, m) is in exactly one of the turnabout sets $T(p_1)$, $T(p_2)$, or $T(p_3)$. Set $n = 4$. Then this sum yields $P(k) = 3k - (k - 1) + (n - k) = n + k + 1$. Thus, $P = n + 2 \quad n + 3 \quad n + 4 \quad n + 5 = 6 \ 7 \ 8 \ 9$ and since P's terms are increasing, its turnabout set is empty and its turnabout number is 0.

2nd solution: Since all pairs appear exactly once in a turnabout set of p_m, all pairs will contribute to the computation of $P(k) = p_1(k) + p_2(k) + p_3(k)$ as described in #7. The set of all pairs is $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$. Thus,

		sum of 3 k's	+	# of times k is first	–	# of times k is last	=	Total
$P(1) = p_1(1) + p_2(1) + p_3(1)$	=	1+1+1	+	3	–	0	=	6
$P(2) = p_1(2) + p_2(2) + p_3(2)$	=	2+2+2	+	2	–	1	=	7
$P(3) = p_1(3) + p_2(3) + p_3(3)$	=	3+3+3	+	1	–	2	=	8
$P(4) = p_1(4) + p_2(4) + p_3(4)$	=	4+4+4	+	0	–	3	=	9

Thus, $P = 6\ 7\ 8\ 9$, P's elements are increasing and the turnabout set of $P = \varnothing$.

10. #1: Suppose $p_1, p_2, \ldots, p_m$ are m permutations of $\{1, 2, \ldots, n\}$ so that every pair (i, j) appears the same number of times ℓ in the set of turnabout sets $T(p_1), T(p_2), \ldots, T(p_m)$. Then $P = p_1 + \ldots + p_m$. Then the permutation $P = p_1 + \ldots + p_m$ is given by $P(k) = mk + \ell(n - 2k + 1)$.

#2: If p_1 through p_{n-1} are permutations of $\{1, 2, \ldots, n\}$ for $n \geq 4$ so that every pair appears exactly once in a turnabout set of p for some m, then $P(k) = (k + 1)n - (3k - 1)$ where $P = p_1 + \ldots + p_n$ and the turnabout set of P is empty.

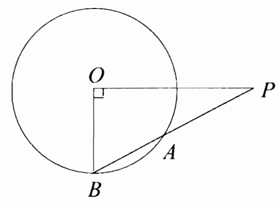
Proof: The set of all pairs is $\{(1, 2), \ldots, (1, n), (2, 3), \ldots, (2, n), \ldots, (n-1, n)\}$. Proceeding as in #9 we have:

$$P(1) = 1(n-1) + (n-1) - 0 = 2n - 2$$
$$P(2) = 2(n-1) + (n-2) - 1 = 3n - 5$$
$$P(k) = k(n-1) + (n-k) - (k-1) = (k+1)n - (3k-1)$$
$$P(n) = n(n+1) - (3n-1) = (n-1)^2$$

Then $P = 2n–2\quad 3n–5\ \ .\ .\ .\ \ (k+1)n - (3k-1)\ \ .\ .\ \ (n-1)^2$. We can show that P is increasing and thus has no pairs in its turnabout set by considering the difference of successive terms: $P(k) - P(k-1) = (k+1)n - (3k-1) - \big(kn - (3k-4)\big)$ which equals $n - 3$. This is greater than 0 for $n \geq 4$. Thus, for $n \geq 4$, $T = \varnothing$ and $N = 0$.

I–1. A pro athlete played for 17 years and earned 72 million dollars. She was paid k million a year where k is an integer and received an extra one million each year that her team made the playoffs. Compute the number of years her team made the playoffs.

I–2. In circle O, $\overline{PO} \perp \overline{OB}$, and PO equals the length of the diameter of circle O. Compute $\dfrac{PA}{AB}$.



I–3. If a and b are the roots of $11x^2 - 4x - 2 = 0$, then compute the product:

$$\left(1 + a + a^2 + a^3 + \ldots\right)\left(1 + b + b^2 + b^3 + \ldots\right)$$

I–4. The measure of $\angle PAQ$ is 60°, $\overrightarrow{AB}$ bisects $\angle PAQ$ and circles P and Q are tangent to $\overrightarrow{AB}$. If the radii of circles P and Q are 1 and 2 respectively, compute the distance from P to Q.

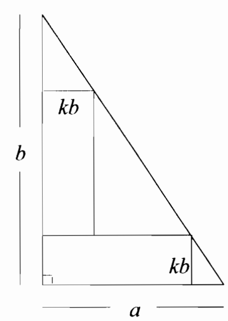
I–5. If $\dfrac{\log_b a}{\log_c a} = \dfrac{19}{99}$, then $\dfrac{b}{c} = c^k$. Compute k.

I–6. Let $A_0A_1 \ldots A_{11}$ be a regular 12–sided polygon whose area is 24. Form convex polygons by first connecting consecutive vertices, then joining the first and the last vertices. For example, polygons $A_3A_4A_5A_6$ or $A_{10}A_{11}A_0A_1A_2A_3A_4$, as well as the entire 12-gon can be formed. Compute the sum of the areas of all the distinct polygons that can be formed.

I–7. Arrange the following products in increasing order from left to right:

1000! (400!)(400!)(200!) (500!)(500!) (600!)(300!)(100!) (700!)(300!)

I–8. Two congruent rectangles are positioned in a right triangle of legs a and b as shown. Both rectangles have a vertex on the hypotenuse and a long side on a leg. If a and b can vary and the short side of each rectangle is kb for real numbers k, compute the largest possible value for k.



1. 4

2. $\frac{3}{2}$

3. $\frac{11}{5} = 2.2$

4. $2\sqrt{3}$

5. $\frac{80}{19}$

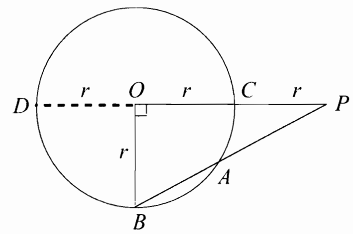
6. 1320

7. $400!400!200! < 600!300!100! < 500!500! < 700!300! < 1000!$

8. $\frac{1}{5}$

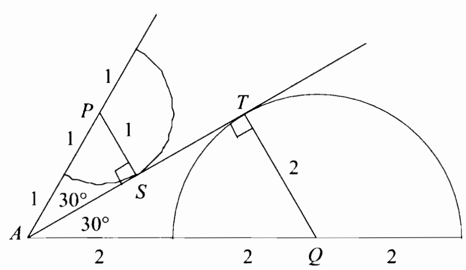
I–1. $17k + n = 72 \rightarrow k = \dfrac{72 - n}{17}$ for $0 \le n \le 17$. The only solution is $n = \boxed{4}$.

I–2. Let $PO = 2r$, $OB = r$, making $PB = r \cdot \sqrt{5}$. Since $PC \cdot PD = PA \cdot PB$, then $r(3r) = PA \cdot \left(r\sqrt{5}\right)$. Then $PA = \dfrac{3r}{\sqrt{5}}$ making $AB = r\sqrt{5} - \dfrac{3r}{\sqrt{5}} = \dfrac{2r}{\sqrt{5}}$. Thus, $\dfrac{PA}{AB} = \boxed{\dfrac{3}{2}}$.



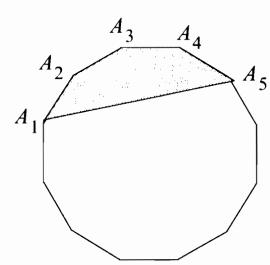
I–3. The roots are $\dfrac{4 \pm \sqrt{104}}{22}$ and both are between –1 and 1. Then both $1 + a + a^2 + a^3 + \ldots$ and $1 + b + b^2 + b^3 + \ldots$ are converging geometric series equaling $\dfrac{1}{1-a}$ and $\dfrac{1}{1-b}$ respectively. Their product equals $\dfrac{1}{1-(a+b)+ab}$ where $(a + b)$ and ab are the sum and product respectively of the roots of $x^2 - \dfrac{4}{11}x - \dfrac{2}{11} = 0$, making $\dfrac{1}{1-(a+b)+ab} = \dfrac{1}{1 - \frac{4}{11} - \frac{2}{11}} = \boxed{\dfrac{11}{5} = 2.2}$.

I–4. Using the diagram at the right, ΔASP and ΔATQ are 30-60-90 triangles, making $AP = 2$ and $AQ = 4$. ΔPAQ has sides 2, 4, and an included angle of 60°. Thus, by the Law of Cosines $PQ^2 = 2^2 + 4\sqrt{2} - 2 \cdot 2 \cdot 4\cos 60° = 12$, making $PQ = \boxed{2\sqrt{3}} = AT$.



I–5. Set $\log_b a = 19t$ and $\log_c a = 99t$, giving $a = b^{19t}$ and $a = c^{99t} \rightarrow b^{19t} = c^{99t} \rightarrow b^{19} = c^{99}$. Thus, $\dfrac{b^{19}}{c^{19}} = c^{80} \rightarrow \dfrac{b}{c} = c^{80/19}$. Thus, $k = \boxed{\dfrac{80}{19}}$.

I–6. Shown is convex polygon $A_1A_2A_3A_4A_5$. It can be paired with the unshaded polygon. So every polygon is in a 1-to-1 correspondence with the complement of that polygon's interior in the 12-gon and the sum of the areas of each pair is the area of the 12-gon, namely 24. A pair of polygons is determined by the border between them. For example, border $\overline{A_1A_5}$



determines the two polygons in the figure at the right. There are ${}_{12}C_2 = 66$ such pairs, but of the 12 pairs of consecutive vertices A_iA_{i+1} which generate the entire 12-gon, we can use only 1, so the sum of all the areas is $(66-11)\cdot 24 = \boxed{1320}$.

I–7. Since ${}_{1000}C_{300} > 1$, $\dfrac{1000!}{700!\cdot 300!} > 1 \to 1000! > 700!\cdot 300!$.

Since ${}_{1000}C_{500} > {}_{1000}C_{300}$, then $\dfrac{1000!}{500!\cdot 500!} > \dfrac{1000!}{700!\cdot 300!} \to 700!\cdot 300! > 500!\cdot 500!$.

Since $\dfrac{500!\cdot 500!}{600!\cdot 300!\cdot 100!} = \dfrac{500!\cdot(500\cdot 499\cdot\ldots\cdot 301)\cdot 300!}{(600\cdot 599\cdot\ldots\cdot 501)\cdot 500!\cdot 300!\cdot 100!} = \dfrac{(500\cdot\ldots\cdot 401)(400\cdot\ldots\cdot 301)}{(600\cdot\ldots\cdot 501)(100\cdot\ldots\cdot 1)}$ and since

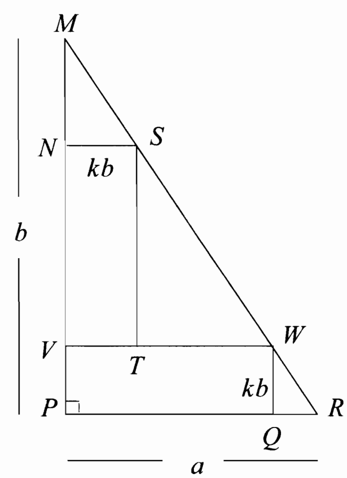
$\dfrac{(500\cdot\ldots\cdot 401)(400\cdot\ldots\cdot 301)}{(600\cdot\ldots\cdot 501)(100\cdot\ldots\cdot 1)} > \left(\dfrac{4}{5}\right)^{100}\cdot\left(\dfrac{3}{1}\right)^{100} = \left(\dfrac{12}{5}\right)^{100} > 1$, we have

$500!\cdot 500! > 600!\cdot 300!\cdot 100!$.

To compare $400!\cdot 400!\cdot 200!$ and $600!\cdot 300!\cdot 100!$, divide both by $400!\cdot 300!\cdot 100!$, obtaining $(400\cdot\ldots\cdot 301)(200\cdot\ldots\cdot 101)$ and $600\cdot 599\cdot\ldots\cdot 402\cdot 401$ respectively. Both contain 200 terms, but clearly the expression on the right is greater on a term by term basis. Thus,

$\boxed{400!\cdot 400!\cdot 200! < 600!\cdot 300!\cdot 100! < 500!\cdot 500! < 700!\cdot 300! < 1000!}$.

I–8. Since $\Delta WQR \sim \Delta MPR$, then $\dfrac{WQ}{QR} = \dfrac{b}{a}$, making $QR = ka$. Thus, $PQ = a - ka = NV$. Since $\Delta MNS \sim \Delta MPR$, $\dfrac{MN}{NS} = \dfrac{b}{a}$, making $MN = \dfrac{kb^2}{a}$. Since $PV + VN + NM = b$, we have $kb + (a - ak) + \dfrac{kb^2}{a} = b$, making $k = \dfrac{ab - a^2}{ab - a^2 + b^2}$. Consider



$\dfrac{1}{k} = \dfrac{ab - a^2 + b^2}{ab - a^2} = 1 + \dfrac{b^2}{a(b-a)}$. Let $t = b - a$. Then

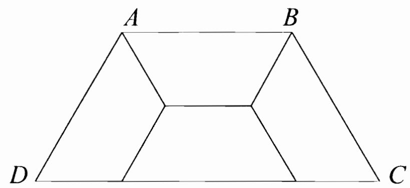
$\dfrac{1}{k} = 1 + \dfrac{(a+t)^2}{at} = 1 + \dfrac{a}{t} + 2 + \dfrac{t}{a}$. Since $\dfrac{a}{t} + \dfrac{t}{a} \ge 2$, then

$\dfrac{1}{k} \ge 1 + 2 + 2 \to \dfrac{1}{k} \ge 5$, making $k \le \dfrac{1}{5}$. Ans: $\boxed{\dfrac{1}{5}}$.

ARML Relay #1 – 1999

R1–1. The sum of the digits of the year 1999 is 28. Let Y be the next following year in which the sum of the digits is 28. Compute $Y - 1999$.

R1–2. Let T = TNYWR and set $k = \dfrac{T}{225}$. Trapezoid $ABCD$ is divided into four congruent trapezoids as shown. If $AB = k$ and $DC = 2k$, compute the sum of the lengths of all line segments in the figure.

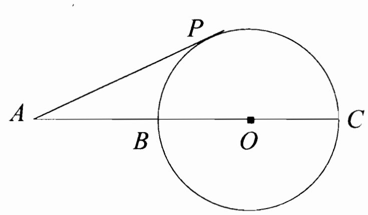


R1–3. Let T = TNYWR. In ΔABC, $\sin\angle A = \cos\angle B$. If, for x in degrees, $m\angle A = x^2 + 5x$, $m\angle B = x^2 - x - 36$, and $|m\angle A - m\angle B| = T - 12$, compute $m\angle A$.

ARML Relay #2 – 1999

R2–1. Let k be the smallest of six consecutive positive integers. If the sum of the six integers is divisible by three distinct primes, compute the smallest possible value for k.

R2–2. Let T = TNYWR. $\overline{BC}$ is a diameter of circle O and $\overline{AP}$ is tangent to circle O at P. If $AP = 2T$ and $AB = T$, compute the length of $\overline{AO}$.



R2–3. Let T = TNYWR. For x randomly chosen from the interval $\left[0, \dfrac{2T}{25}\right]$, compute the probability that $15x^2 + 3 < 14x$.

Relay #1:

R1–1. 900

R1–2. 30

R1–3. 36

Relay #2:

R2–1. 15

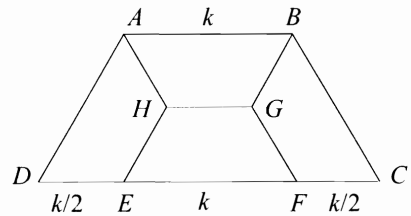
R2–2. $\dfrac{75}{2}$

R2–3. $\dfrac{4}{45}$

Solutions to ARML Relay #1 – 1999

R1–1. For the year $2ABC$, $A + B + C = 26$. This can only be done using one eight and two nines. To minimize, choose the eight for the hundred's place giving 2899. Then $2899 - 1999 = \boxed{900}$.

R1–2. Since $AB = k$, then $AD = BC = EF = k$, but if $EF = k$, then $DE = FC = \frac{k}{2}$. Thus, the legs of the four congruent trapezoids are all $\frac{k}{2}$, and since legs are also bases, $EH = HG = GF = AH = GB = \frac{k}{2}$. The sum of the segments is $2k + 3k + 5\left(\frac{k}{2}\right) = \frac{15k}{2}$. Since $k = \frac{900}{225} = 4$, the sum is $\boxed{30}$.



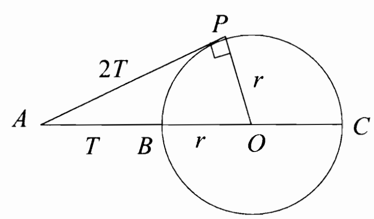
R1–3. If $\sin A = \cos B$, then either $A = 90° - B$ or $A = 90° + B$. Using $A = 90° - B$, then $\left(x^2 + 5x\right) + \left(x^2 - x - 36\right) = 90° \rightarrow x^2 + 2x - 63 = 0 \rightarrow x = 7$ or -9. If $x = 7, A = 84°$ and $B = 6°$. If $x = -9$, then $A = 36°$ and $B = 54°$. Using $A = 90° + B$, then $\left(x^2 + 5x\right) = 90° + \left(x^2 - x - 36\right) \rightarrow 6x = 54 \rightarrow x = 9 \rightarrow A = 126°$ and $B = 36°$.

Since $T = 30$, then $|m\angle A - m\angle B| = 18°$ and only $A = 36°, B = 54°$ works. Ans: $\boxed{36}$.

Solutions to ARML Relay #2 – 1999

R2–1. $k + (k+1) + \ldots + (k+5) = 6k + 15 = 3(2k+5)$. Since $2k + 5$ is odd, 2 is not one of the primes. The next smallest primes are 5 and 7, so set $2k + 5 = 5 \cdot 7 \rightarrow k = \boxed{15}$.

R2–2. $T = 15$. Let $OB = r \rightarrow (2T)^2 + r^2 = (T + r)^2$ implies $3T = 2r \rightarrow r = \frac{3T}{2}$. Thus, $AO = T + r = \frac{5T}{2} = \boxed{\frac{75}{2}}$.



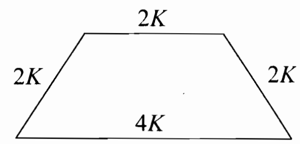
R2–3. $T = \frac{75}{2}$ so $\frac{2T}{25} = 3$. Thus, $15x^2 - 14x + 3 < 0 \rightarrow (5x-3)(3x-1) < 0 \rightarrow \frac{1}{3} < x < \frac{3}{5}$. Since $\frac{3}{5} - \frac{1}{3} = \frac{4}{15}$, the probability is $\frac{4/15}{3} = \boxed{\frac{4}{45}}$.

Note: Pass answers from position 1 to position 15.

1. Let $N = 3A4A5A$ where A is a digit chosen at random from $\{0,1,2,\ldots,9\}$. Compute the probability that N is divisible by 9.

2. Let T = TNYWR and set $K = 20T$. Compute $(1+i)^{2K}$.

3. Let T = TNYWR and set $K = \dfrac{T}{-16}$. If the area of the trapezoid at the right equals A, pass back $A\sqrt{3}$.



4. Let T = TNYWR. If $\sin x = T\cos x$, compute the value of $(\sec x - T)(\sec x + T)$.

5. Let T = TNYWR and set $K = 14T$. A convex polygon has K sides. Compute the number of diagonals of the polygon.

6. Let T = TNYWR. Let p be a prime number and b be a positive integer. If $\log b = T\log p$, then b has X more factors than p does. Compute X.

7. Let T = TNYWR and set $K = \dfrac{T}{4}$. If $x^2 + 2x + 4 = \dfrac{K}{x-2}$, compute x.

8. Let T = TNYWR. The vertex of a concave up parabola is $V(T, -16)$. Compute the sum of the zeros of the parabola.

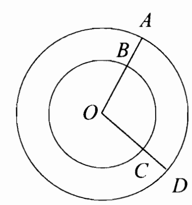
9. Let T = TNYWR and set $K = 100T$. Let a_{n+1} = the sum of the cubes of the digits of a_n for $n \geq 1$. If $a_1 = 22$, compute a_K.

10. Let T = TNYWR. Given the points $A(\log 2, \log 3)$ and $B\left(\log\left(\log T^2\right), \log\left(\log T^3\right)\right)$, compute the slope of $\overleftrightarrow{AB}$.

11. Let T = TNYWR. The intercepts of the lines $x + y = T$ and $x + y = T + 1$ form a trapezoid. Compute the area of the trapezoid.

12. Let T = TNYWR and set $K = 2T + 4$. Compute the value of x which solves $\sqrt{K - 2x} = 2x - 1$.

13. Let T = TNYWR. The two circles are concentric. If the ratio of the radius of the large circle to the radius of the small circle equals T, compute the ratio of the area of partial ring $ABCD$ to the area of sector BOC.



14. Let T = TNYWR and set $K = 4T$. ΔABC is a right triangle and the radius of its inscribed circle equals K. If the perimeter of ΔABC exceeds twice its hypotenuse by N, compute N.

15. Let T = TNYWR. Children are equally spaced about a circle. If a child in position T is directly opposite a child in position 58, compute the number of children who are equally spaced about the circle.

1. $\frac{3}{10}$

2. –64

3. 144

4. 1

5. 77

6. 76

7. 3

8. 6

9. 217

10. 1

11. $\frac{3}{2}$

12. $\frac{3}{2}$

13. $\frac{5}{4}$

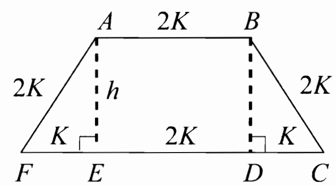
14. 10

15. 96

1. $3A + 12 = 3(A + 4)$ must be divisible by 9 $\rightarrow$ $A + 4$ must be a multiple of 3. Thus, $A = 2, 5$, or 8 and the probability is $\boxed{\dfrac{3}{10}}$.

2. $\left((1+i)^2\right)^K = (2i)^K$. Since $T = \dfrac{3}{10}$, $K = 6$ and $(2i)^6 = \boxed{-64}$.

3. Since $AF = 2FE$, ΔAFE and ΔBCD are 30-60-90 triangles, making $AE = K\sqrt{3}$. The area of the trapezoid $= \left(\dfrac{1}{2}\right)\left(K\sqrt{3}\right)\left(2K + 4K\right) = 3K^2\sqrt{3}$. Since $T = -64$, $K = 4$, so the area $= 48\sqrt{3}$. Pass back $48\sqrt{3}\sqrt{3} = \boxed{144}$.



4. $\sin x = T\cos x \rightarrow \tan x = T \rightarrow 1 + \tan^2 x = 1 + T^2 = \sec^2 x$. Thus, $\sec^2 x - T^2 = \boxed{1}$. Note that the value of T was irrelevant in this problem.

5. For each vertex in a polygon of K sides there are $K - 3$ diagonals. Hence, the total number of diagonals equals $\dfrac{K(K-3)}{2}$. Since $K = 14$, the result is $\boxed{77}$.

6. Since $b = p^T$ and p has two factors, 1 and p, then b has the following $T + 1$ factors: $1, p, p^2, \ldots, p^T$. Thus, $X = (T + 1) - 2 = T - 1$. Since $T = 77$, pass back $\boxed{76}$.

7. $\left(x^2 + 2x + 4\right)\left(x - 2\right) = K \rightarrow x^3 - 8 = K \rightarrow x^3 = K + 8$. Since $T = 76$, $K = 19 \rightarrow x^3 = 27 \rightarrow x = \boxed{3}$.

8. Equation of the parabola: $y = a(x - T)^2 - 16$. If $y = 0$, then $x = T \pm \dfrac{4}{\sqrt{a}}$. The sum of the two zeros equals $2T$. Since $T = 3$, pass back $\boxed{6}$.

9. Since $a_1 = 22$, $a_2 = 16$, $a_3 = 217$, $a_4 = 352$, $a_5 = 160$, and $a_6 = 217$, the sequence is cyclic with $a_{3n} = 217$, $a_{3n+1} = 352$, and $a_{3n+2} = 160$. Since $T = 6$, $K = 600$ and $a_{600} = \boxed{217}$.

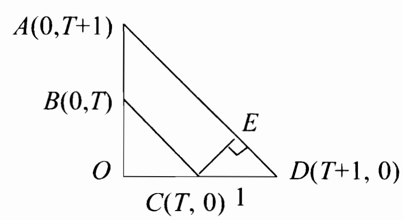
10. Note that the domain of $\log\left(\log T^3\right)$ consists of all T for which $\log T > 0 \rightarrow T > 1$. All these values work for $\log\left(\log T^2\right)$. Slope of $\overleftrightarrow{AB} = \dfrac{\log\left(\log T^3\right) - \log 3}{\log\left(\log T^2\right) - \log 2} = \dfrac{\log\left(\dfrac{\log T^3}{3}\right)}{\log\left(\dfrac{\log T^2}{2}\right)} = \dfrac{\log\left(\dfrac{3\log T}{3}\right)}{\log\left(\dfrac{2\log T}{2}\right)} = \dfrac{\log(\log T)}{\log(\log T)} = \boxed{1}$.

Note that as long as $T > 1$, then the value of T is irrelevant

11. Area of trapezoid $ABCD = a(\Delta AOD) - a(\Delta BOC) =$

$\frac{1}{2}(T+1)^2 - \frac{1}{2} \cdot T^2 = \frac{2T+1}{2}$. Since $T = 1$,

the area of $ABCD = \boxed{\frac{3}{2}}$.

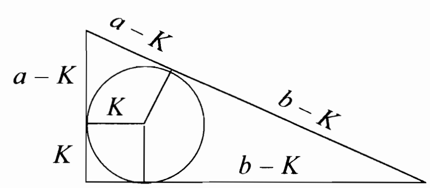


12. $K - 2x = 4x^2 - 4x + 1 \rightarrow 4x^2 - 2x + (1 - K) = 0 \rightarrow x = \dfrac{2 \pm \sqrt{4 - 4 \cdot 4(1-K)}}{8} = \dfrac{1 \pm \sqrt{4K - 3}}{4}$. Try 'nice' values for K: if $K = 1, x = \frac{1}{2}$, if $K = 3, x = 1$, etc. Since $T = \frac{3}{2}, K = 7$, making $x = \boxed{\frac{3}{2}}$.

13. If $T = \frac{m}{n}$, then $\dfrac{a(\text{sector } AOD)}{a(\text{sector } BOC)} = \dfrac{m^2}{n^2}$. The ratio of the partial ring $ABCD$ to the sector BOC is $\dfrac{m^2 - n^2}{n^2}$.

Since $T = \frac{3}{2}, m = 3, n = 2$, giving $\boxed{\frac{5}{4}}$.

14. Let a and b be the lengths of the legs and c be the length of the hypotenuse. Since $c = a - K + b - K$, then $2K = a + b - c$ giving $2K + 2c = a + b + c$. Thus, $N = 2K$. Since $T = \frac{5}{4}$, $K = 5$, making $N = \boxed{10}$.



15. Two cases: if $T < 58$, then there are $58 - T - 1$ children on each side of the circle between T and 58. The total is $2(58 - T - 1) + 2 = 116 - 2T$ for $T \leq 29$. If $T > 58$, then there are $2(T - 58 - 1) + 2 = 2T - 116$ for $T \geq 116$. Since $T = 10$, the number of children is $116 - 2 \cdot 10 = \boxed{96}$.

1. Let $f(x) = (x+3)^2 + \frac{9}{4}$ for $x \geq -3$. Compute the shortest possible distance between a point on f and a point on f^{-1}.

1. The shortest distance will connect the points $(x, f(x))$ and $(f(x), x)$ since each is the reflection of the other across $y = x$. Let $d = \sqrt{(x - f(x))^2 + (f(x) - x)^2} = |f(x) - x| \cdot \sqrt{2}$. To minimize d we must minimize $|f(x) - x| = \left|x^2 + 5x + \frac{45}{4}\right|$. For the general quadratic function $y = ax^2 + bx + c$ the minimum occurs at $x = \frac{-b}{2a}$. In this case the minimum occurs at $x = \frac{-5}{2}$, and substituting, we obtain a minimum of 5.

Thus, $d = \boxed{5\sqrt{2}}$.

ARML

2000

ARML celebrated its 25th annual competition in grand style with a real nail-biter of a contest. There were 104 teams with 1560 students taking part at Penn State, the University of Iowa, and UNLV. For the first time ever there was a tie for first place as Chicago A and San Francisco Bay A each scored 172 points, just one more than third place New York City A. But there was also a tie in Division B, the first since 1983. Both Connecticut A and Peninsula South Bay scored 127 points, just two ahead of Iowa A and Northern California. Three teams from Taiwan took part in this year's contest and did quite well. Taiwan's parallel contest, TRML, involved 80 teams last year and this year they are expecting some 280. Talk about explosive growth. Mark Saul, president of ARML, was feted at a banquet and given a Lifetime Achievement Award in honor of his long and effective service. Chris Clark, formerly of the Western Massachusetts team, was in charge of organizing special events in honor of ARML's 25th year and he came up will all sorts of good ideas, including a lovely folder with the ARML logo on the front, and a collection of the best of the ARML problems over the years. The following problem was written in honor of ARML's 25th anniversary and students enjoyed solving it:

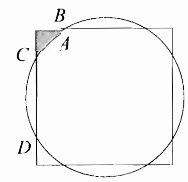
> The decimal $.ARML_b$ equals $.\overline{25}_{10}$. If b is a positive integer that is as small as possible, find the sum of $A + R + M + L$.

Debbie Clyde of Idaho received the Samuel Greitzer Distinguished Coach Award.

Richard Rukin who was instrumental in getting a second ARML site started at the University of Iowa received the Alfred Kalfus Founder's Award.

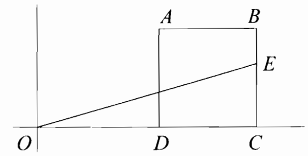
Andy Yang of Howard County won the Zachary Sobol Award for outstanding contributions to his ARML team.

T–1. In the diagram, the circle and square have the same center. If the area of shaded region ABC equals the area of the region bounded by $\overline{CD}$ and minor arc $\overset{\frown}{CD}$, compute the ratio of the side of the square to the radius of the circle.



T–2. Compute the smallest prime number p such that $p^3 + 2p^2 + p$ has exactly 42 positive factors.

T–3. Square $ABCD$ is divided into two regions of equal area by $\overline{OE}$. If OD and AD are integers and $\dfrac{CE}{BE} = 2000$, compute the smallest value of AD.



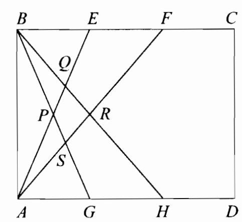
T–4. In base 10, in the equation TWO + TWO = FOUR, distinct letters represent different digits and F $\neq$ 0. Compute the smallest possible value for FOUR.

T–5. Equilateral triangle ABC has sides of length 6 and its medians, AA', BB', and CC', intersect at D. If three segments are chosen from among AD, $A'D$, BD, $B'D$, CD and $C'D$ to form a non-equilateral triangle of positive area, compute the area of that triangle.

T–6. Let x be a randomly selected integer from the set $\{100, 101, \ldots, 999\}$. What is the probability that x^2 and $(x + 100)^2$ have the same number of digits?

T–7. Let $f(x) = (x - 1)(x - 2)^2(x - 3)^3 \cdots (x - 1999)^{1999}(x - 2000)^{2000}$. Compute the number of values of x for which $|f(x)| = 1$.

T–8. In rectangle $ABCD$, G and H are trisection points of $\overline{AD}$, and E and F are trisection points of $\overline{BC}$. If $AB = 360$ and $BC = 450$, compute the area of $PQRS$.



T–9. For an integer k in base 10, let $z(k)$ be the number of zeros that appear in the binary representation of k. Let $S_n = \displaystyle\sum_{k=1}^{n} z(k)$. Compute S_{256}.

T–10. The unit hypercube in 8 dimensions can be defined as the set of points $(x_1, x_2, \ldots, x_8)$ such that $0 \le x_i \le 1$ for $i \in \{1, 2, \ldots, 8\}$. Suppose the hypercube is cut into smaller hyper-rectangular boxes with side lengths of $\frac{1}{3}$ or $\frac{2}{3}$ by hyperplanes whose equations are $x_i = \frac{1}{3}$ for $i \in \{1, 2, \ldots, 8\}$. Color the hyper-rectangular box with vertex on the origin red, color those adjacent to red hyper-rectangular boxes blue, and those adjacent to blue hyper-rectangular boxes red where "adjacent" means bordering along one of the hyperplanes $x_i = \frac{1}{3}$. If R equals the total volume of the red hyper-rectangular boxes and B equals the total volume of the blue hyper-rectangular boxes, compute $R - B$.

1. $\sqrt{\pi}$ or $\sqrt{\pi}:1$

2. 23

3. 1999

4. 1468

5. $\dfrac{3\sqrt{15}}{4}$

6. $\dfrac{7}{9}$

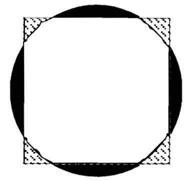
7. 4000

8. 4500

9. 777

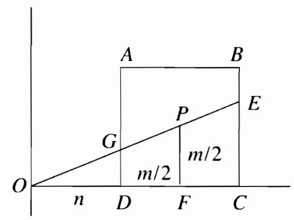
10. $\dfrac{1}{6561}$

T–1. Since the sum of the areas of the dark-shaded regions equals the sum of the areas of the light-shaded regions, the area of the square equals the area of the circle. Thus, $s^2 = \pi r^2 \rightarrow s : r = \boxed{\sqrt{\pi}}$ or $\boxed{\sqrt{\pi} : 1}$.



T–2. If $p^3 + 2p^2 + p = p(p+1)^2$ has 42 factors when p is prime, then $(p+1)^2$ must have 21 factors. Let $a, b, c, \ldots$ be the exponents of primes in the prime factorization of $p + 1$. Then $(p+1)^2 = \left(p_1^{\,a} \cdot p_2^{\,b} \cdot p_3^{\,c} \cdots\right)^2$. Thus, $(2a+1)(2b+1)(2c+1)\ldots = 21$. This is only possible when $p + 1$ has precisely two prime factors making $2a + 1 = 3$ and $2b + 1 = 7 \rightarrow a = 1$ and $b = 3$. So $p + 1$ has the form $(p_1 \cdot p_2^{\,3})$. Numbers through 22 fail, but $24 = 2^3 \cdot 3$ works, making $p = \boxed{23}$.

T–3. $\overline{OE}$ passes through P, the center of the square. Let $OD = n$ and $AD = m$.



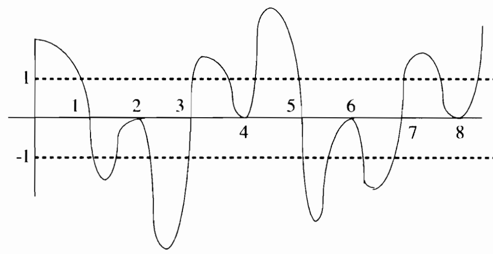
Slope of $\overline{OE} = \dfrac{PF}{FO} = \dfrac{m/2}{n + m/2} = \dfrac{m}{2n+m}$. The equation of $\overline{OE}$ is $y = \dfrac{mx}{2n+m}$. Since $GD = \dfrac{mn}{2n+m}$, $EC = \dfrac{m(n+m)}{2n+m}$, and $GD = BE$, then $\dfrac{EC}{GD} = \dfrac{m(n+m)}{mn} = 2000$. Thus, $n + m = 2000n \rightarrow m = 1999n$. If $n = 1$, then $m = \boxed{1999}$.

T–4. F = 1. Clearly, O $\neq$ 0 since that would make R = 0. Since F = 1 then O $\neq$ 1; try O = 2. This makes T = 6, but no value works for W. Try O = 3. This again makes T = 6 which can't be because R = 6 if O = 3. Try O = 4, making R = 8 and T = 7. W = 3 works $\rightarrow$ TWO + TWO = FOUR = 734 + 734 = $\boxed{1468}$.

T–5. The medians have length $3\sqrt{3}$ and are divided into a ratio of 2 : 1, so we choosing from lengths of $2\sqrt{3}$ and $\sqrt{3}$. We can't pick two of $\sqrt{3}$ and one of $2\sqrt{3}$, so our triangle has sides $2\sqrt{3}$, $2\sqrt{3}$, and $\sqrt{3}$. The triangle is isosceles and the altitude to the base equals $\sqrt{\left(2\sqrt{3}\right)^2 - \left(\sqrt{3}/2\right)^2} = \dfrac{3\sqrt{5}}{2}$. The area is $\boxed{\dfrac{3\sqrt{15}}{4}}$.

T–6. The squares of three-digit numbers have either 5 or 6 digits. If $10{,}000 \le x^2 < 100{,}000$, then $100 \le x \le 316$ and x and $x + 100$ both lie in this range if $x \in \{100, 101, \ldots, 216\}$. If $100{,}000 \le x^2 < 1{,}000{,}000$, then $317 \le x \le 999$ and x and $x + 100$ both lie in this range if $x \in \{317, 318, \ldots, 899\}$. Out of 900 three-digit numbers, there are $(216 - 100 + 1) + (899 - 317 + 1) = 117 + 583 = 700$ values of x satisfying the condition stated in the problem. Probability $= \dfrac{700}{900} = \boxed{\dfrac{7}{9}}$.

T–7. A portion of an approximation to the graph is shown at the right. From 1 to 2000, between every pair of consecutive integers, there are two solutions. There are 2000 – 1 = 1999 pairs. There is 1 solution between 0 and 1 and one solution after 2000, giving a total of $1999 \cdot 2 + 2 = \boxed{4000}$ solutions.

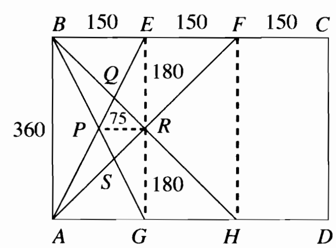


Note: We should verify an assumption of our solution, namely that $|f(x)| > 1$ between every pair of consecutive integers. Note that if n is an integer drawn from $\{1, 2, \ldots, 1999\}$, then $f\left(n + \frac{1}{2}\right)$ is a product of factors that are greater than 1, namely 3/2, 5/2, etc., and factors of 1/2 that arise from the terms $(x - n)^n$ and $(x - (n+1))^{n+1}$. We obtain the most factors of $\frac{1}{2}$ when $n = 1999$. Since $f(1999.5) =$ $(1998.5)^1(1997.5)^2(1996.5)^3 \cdots (2.5)^{1997}(1.5)^{1998}(.5)^{1999}(-.5)^{2000}$, we have a product consisting of $1 + 2 + \ldots + 1998 = \frac{1998 \cdot 1999}{2}$ factors greater than or equal to $\frac{3}{2}$ and 3999 factors of $\frac{1}{2}$. Thus,

$$|f(1999.5)| > (1.5)^{999 \cdot 1999}(.5)^{3999} > (1.5)^{8000}(.5)^{3999} = (2.25)^{4000}(.5)^{3999} > 2^{4000}(.5)^{3999} > 1.$$

Although we didn't expect ARML participants to consider the question of whether $|f(x)|$ takes on a value of 1 more than twice in a given interval [n, n+1], we'll present an answer for completeness. By Rolle's Theorem, $f(x)$ has at least one relative extrema in each interval (n, n+1). Thus, $f(x)$ has <u>at least</u> 1999 relative extrema. Write $f'(x) = f(x)\left(\frac{1}{x-1} + \frac{2}{x-2} + \cdots + \frac{2000}{x-2000}\right)$. Add the fractions to obtain a polynomial in the numerator of degree 1999. Thus, $f'(x)$ has <u>at most</u> 1999 zeros. Therefore, $f(x)$ has <u>exactly</u> 1999 relative extrema, one per interval, and consequently, there are no more than two solutions to $|f(x)| = 1$ in each [n, n+1].

T–8. Since $BF = AH$, $BFHA$ is a rectangle and therefore R is collinear with E and G. Draw $\overline{PR}$ forming congruent trapezoids $PREB$ and $PRGA$. Since $BEGA$ is a rectangle, $PR = \frac{1}{2} \cdot BE = 75$, making the scale factor of similar triangles PQR and EQB equal to $\frac{1}{2}$. The distance from $\overline{PR}$ to $\overline{BE}$ is $\frac{360}{2} = 180$. Set it equal to $3x$, making the distance from Q to $\overline{BE}$ equal to $2x$ and the distance from Q to $\overline{PR}$ equal to x. If $3x = 180$, then $x = 60$. The area of $PQR = \frac{1}{2} \cdot 60 \cdot 75 = 2250$. Thus, the area of $PQRS = \boxed{4500}$.



T–9. Consider this array of 256 eight-digit base 2 numbers written with leading zeros:

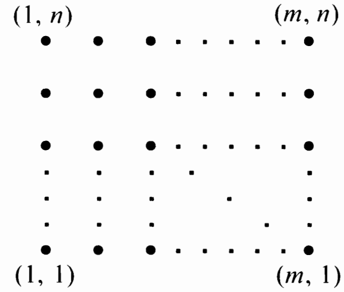
$$\begin{array}{rcl} 0_{10} & = & 00000000_2 \\ 1_{10} & = & 00000001_2 \\ 2_{10} & = & 00000010_2 \\ 3_{10} & = & 00000011_2 \\ . & = & . \\ . & = & . \\ 255_{10} & = & 11111111_2 \end{array}$$

Of the $8 \cdot 256 = 2048$ digits in the array, exactly half, 1024, are zeros. Now we exclude all leading zeros. One number has 8 leading zeros, one has 7, 2 have 6, 4 have 5, 8 have 4, 16 have 3, 32 have 2, 64 have 1, and 128 have none, giving $8 + 7 + 2 \cdot 6 + 4 \cdot 5 + 8 \cdot 4 + 16 \cdot 3 + 32 \cdot 2 + 64 \cdot 1 = 255$. Up to 255_{10} there are $1024 - 255 = 769$ zeros. Add the 8 zeros from $256_{10} = 100000000_2$, making $\boxed{777}$.

Alternate solution: consider, for example, all numbers with a zero in the third position: _ _ 0 _ _ _ _ _ . Using 0's or 1's there are 2^5 numbers that can be formed using the places to the right of 0, and since the number cannot begin with 0, there are $2^2 - 1$ numbers that can be formed using the places to the left of the 0. Thus, there are $\left(2^5\right)\left(2^2 - 1\right)$ numbers with a 0 in the third position. Therefore, for a zero in the ith position there are $\left(2^{8-i}\right)\left(2^{i-1} - 1\right)$ numbers. We can determine the total number of zeros by summing up the number of numbers with a zero in each position. Thus, $\sum_{i=1}^{8}\left(2^{8-i}\right)\left(2^{i-1} - 1\right) = \sum_{i=1}^{8} 2^7 - 2^{8-i} =$ $8 \cdot 2^7 - \left(2^7 + 2^6 + \cdots + 2^1 + 2^0\right) = 8 \cdot 2^7 - \left(2^8 - 1\right) = 3 \cdot 2^8 + 1 = 769$. Now add 8 zeros.

T–10. Let k = the number of edges at each vertex of a given hyper–rectangular box H that have length $\frac{1}{3}$. Then H is red if k is even and blue if k is odd, and the number of given hyper–rectangular boxes with a given value of k is $\binom{8}{k}$. Therefore, $R - B = \sum \binom{8}{k}(-1)^k\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{8-k} = \left(\frac{2}{3} - \frac{1}{3}\right)^8 = \frac{1}{3^8} = \boxed{\frac{1}{6561}}$.

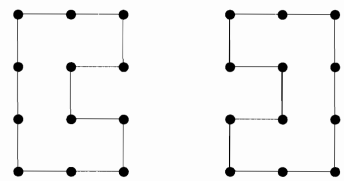
Shown at the right is an m by n array of lattice points. Define an ARML (m, n) polygon to be a simple closed polygon with the following properties:



1. It is constructed by joining all the points in the m by n array with unit line segments that are horizontal or vertical. This can be done without lifting your pencil off the paper.

2. Each point is joined by a unit line segment to exactly two other points.

Two ARML (m, n) polygons are distinct if one joins two points that the other does not. Two distinct ARML (m, n) polygons may be congruent. Define $F(m, n)$ to be the number of distinct ARML (m, n) polygons for a given ordered pair (m, n). For example, the diagram at the right shows all of the distinct ARML (3, 4) polygons, making $F(3, 4) = 2$.



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1. a) On the 6 by 8 grids provided, sketch four non-congruent ARML (6, 8) polygons.

   b) Find the area and perimeter of each polygon.

2. For each of the following values of m and n, compute $F(m, n)$. (You may do so by sketching all of the distinct ARML (m, n) polygons on the grids provided, but do not turn in the grids).

   a) $m = 3, n = 6$ b) $m = 4, n = 4$ c) $m = 4, n = 5$ d) $m = 4, n = 6$

3. a) Compute $F(2, n)$ for all values of $n \geq 2$ and justify your answer.

   b) Let n be an even integer. Find and prove a formula for $F(3, n)$ in terms of n.

4. Using the data from problem #2 and assuming that $F(4, 0) = F(4, 1) = 0$, write a recursive formula for $F(4, n)$ with linear terms and integer coefficients. No proof is required.

5. a) For $m, n \geq 2$ with m and n not both odd, develop a formula for the perimeter of all ARML (m, n) polygons as a function of m and n and prove it.

   b) Prove that if m and n are both odd, $m, n > 2$, then $F(m, n) = 0$.

6. Prove that for every given $m, n \geq 2$ where m and n are not both odd, the area of any ARML (m, n) polygon is given by the formula $\frac{mn}{2} - 1$.

In the following problems let the area of an ARML (m, n) polygon be represented by $K(m, n)$.

For these problems m and n are not both odd and both are greater than or equal to 2.

7. Let x, y, and z be distinct positive integers whose sum exceeds 100. Find an example of three ARML (m, n) polygons of the form $(3, x)$, $(4, y)$, and $(5, z)$ such that $K(3, x)$, $K(4, y)$, and $K(5, z)$ are an increasing arithmetic progression. Show your analysis.

8. Since ARML (3, 14), (4, 11), and (5, 12) polygons have areas of 20, 21, and 29 respectively, their areas form a Pythagorean triple. Determine another set of ARML $(3, x)$, $(4, y)$, and $(5, z)$ polygons whose areas are proportional to a 20–21–29 triple. Show the work that led to the answer.

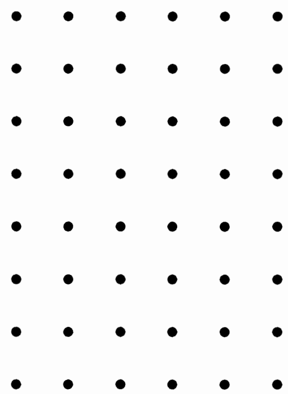
9. a) Determine with proof all ARML (m, n) polygons whose area is half that of an ARML $(m + 7, n + 7)$ polygon.

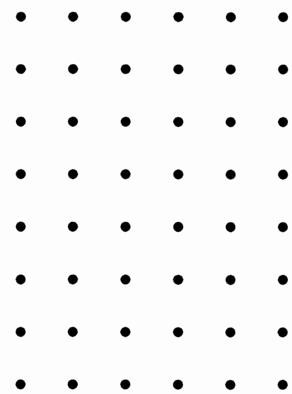
   b) For $n, m \geq 2$, show that there are no values of m and n such that $K(m, m) = \frac{1}{2}K(n, n)$.

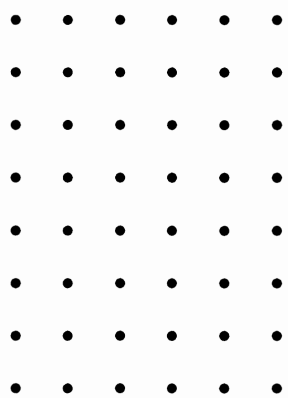
10. For a fixed value k, compute the number of ordered pairs (m, n) such that the area of the ARML $(m + k, n + k)$ polygon is twice that of the ARML (m, n) polygon. Use the prime factorization of $k^2 + 1 = 2^{a_2}3^{a_3}5^{a_5}\cdots$ in your answer.

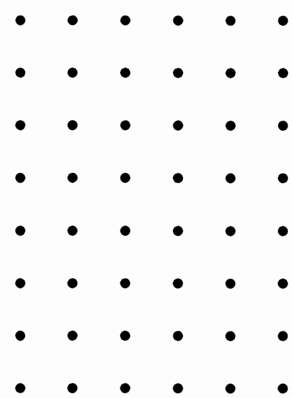
Bonus question: for 4 additional points give a correct proof of the correct formula for problem #4. Do not attempt this problem until you've completely finished the Power Question.

Power Question Answer Sheet: Four ARML (6, 8) Polygons

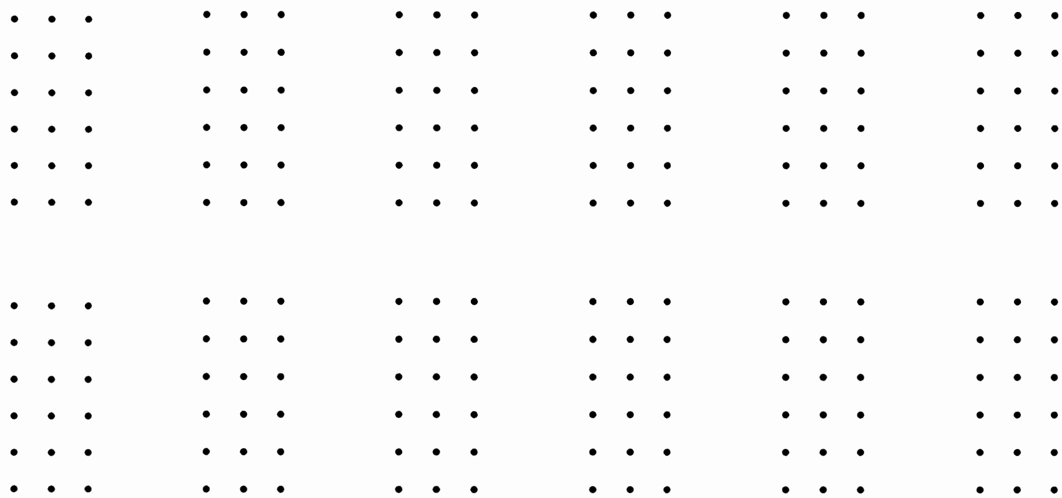




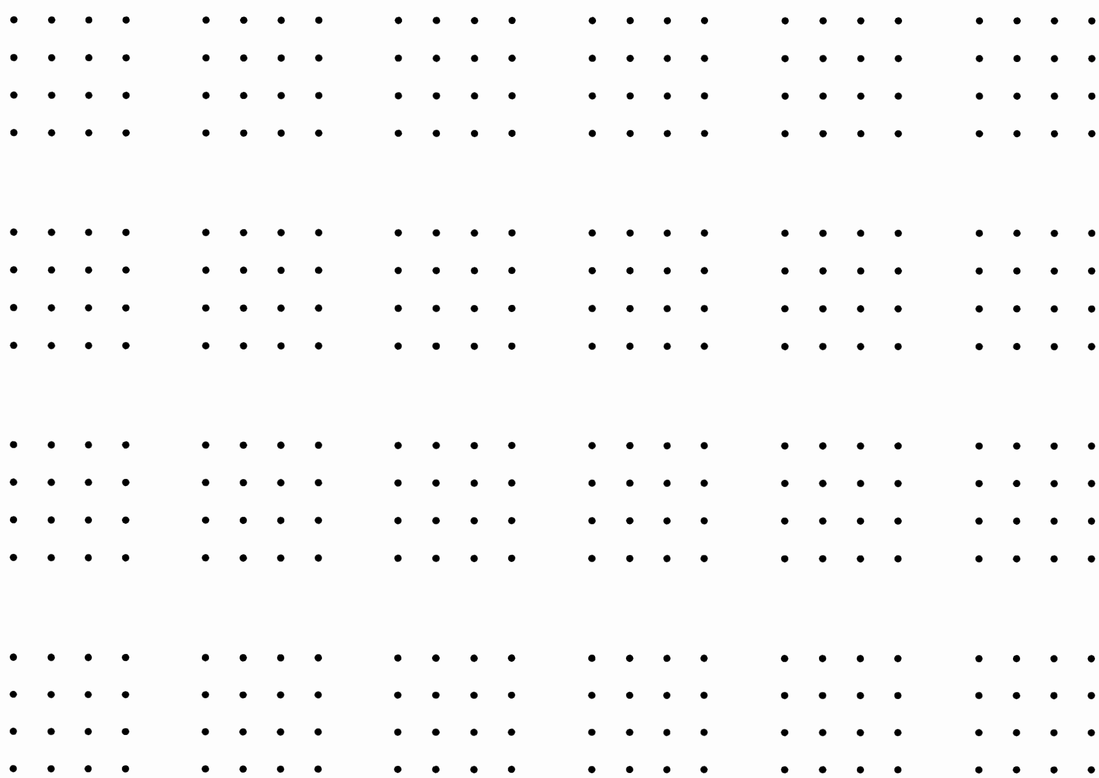




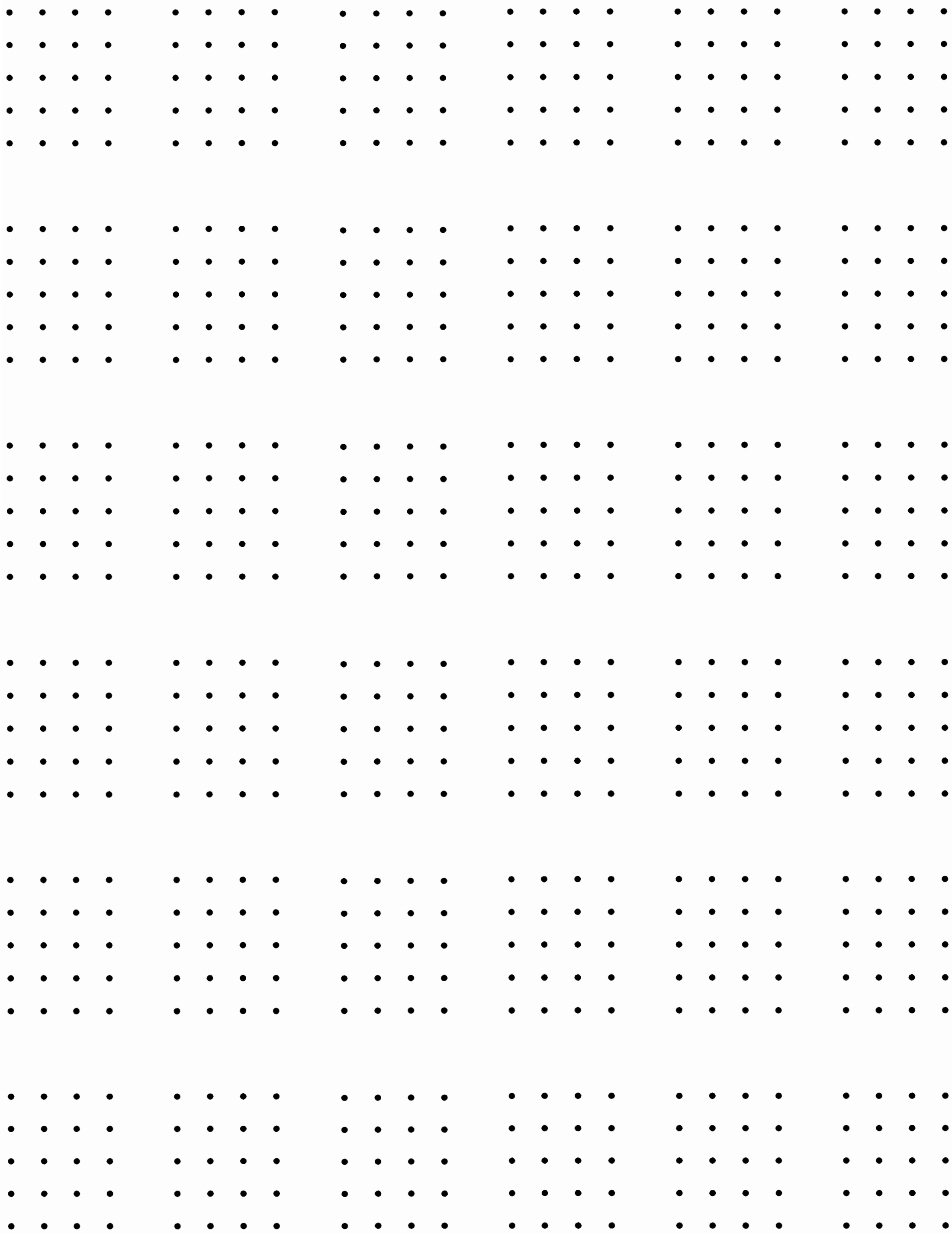
Power Question: ARML (3, 6) Polygons



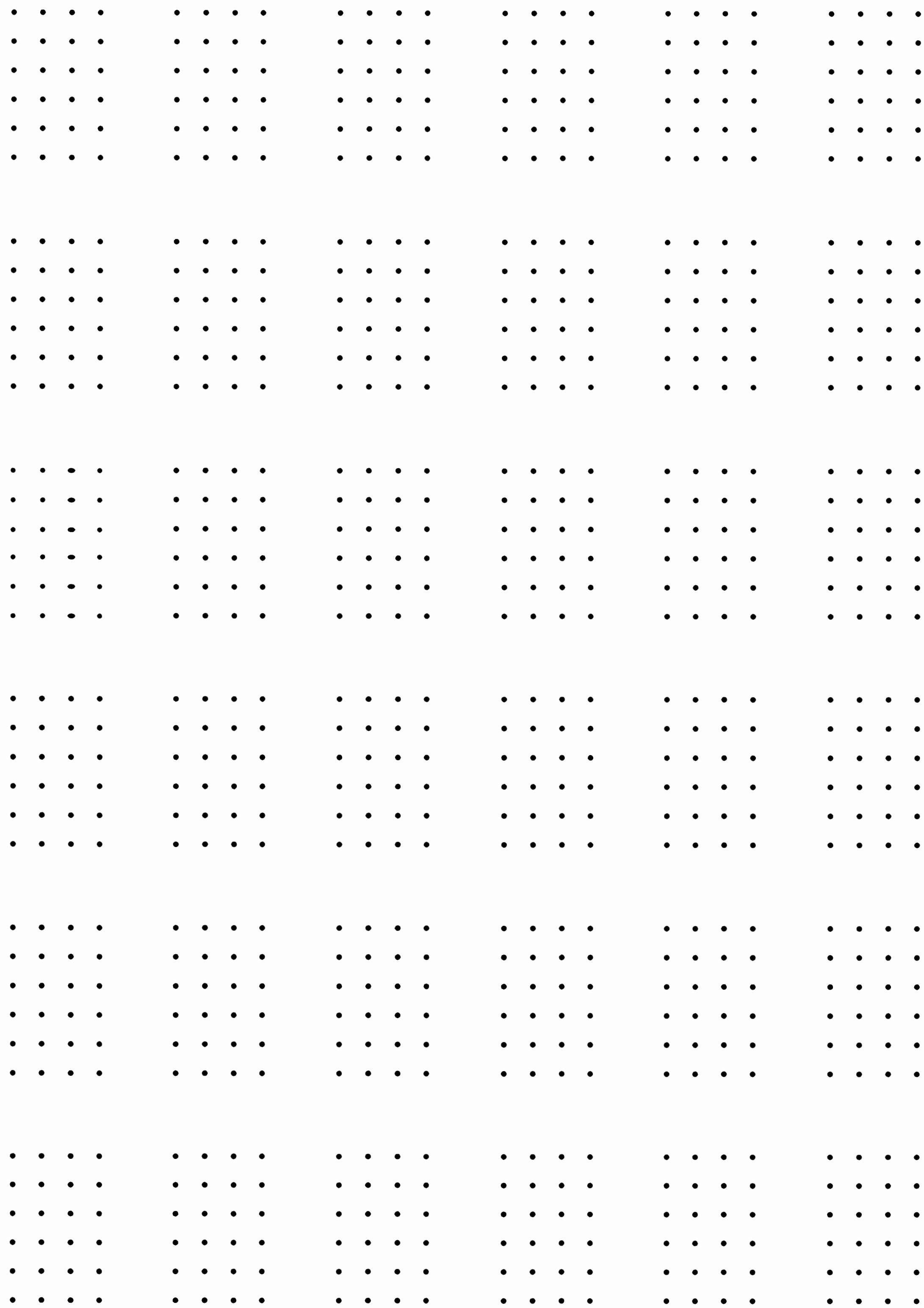
ARML (4, 4) Polygons



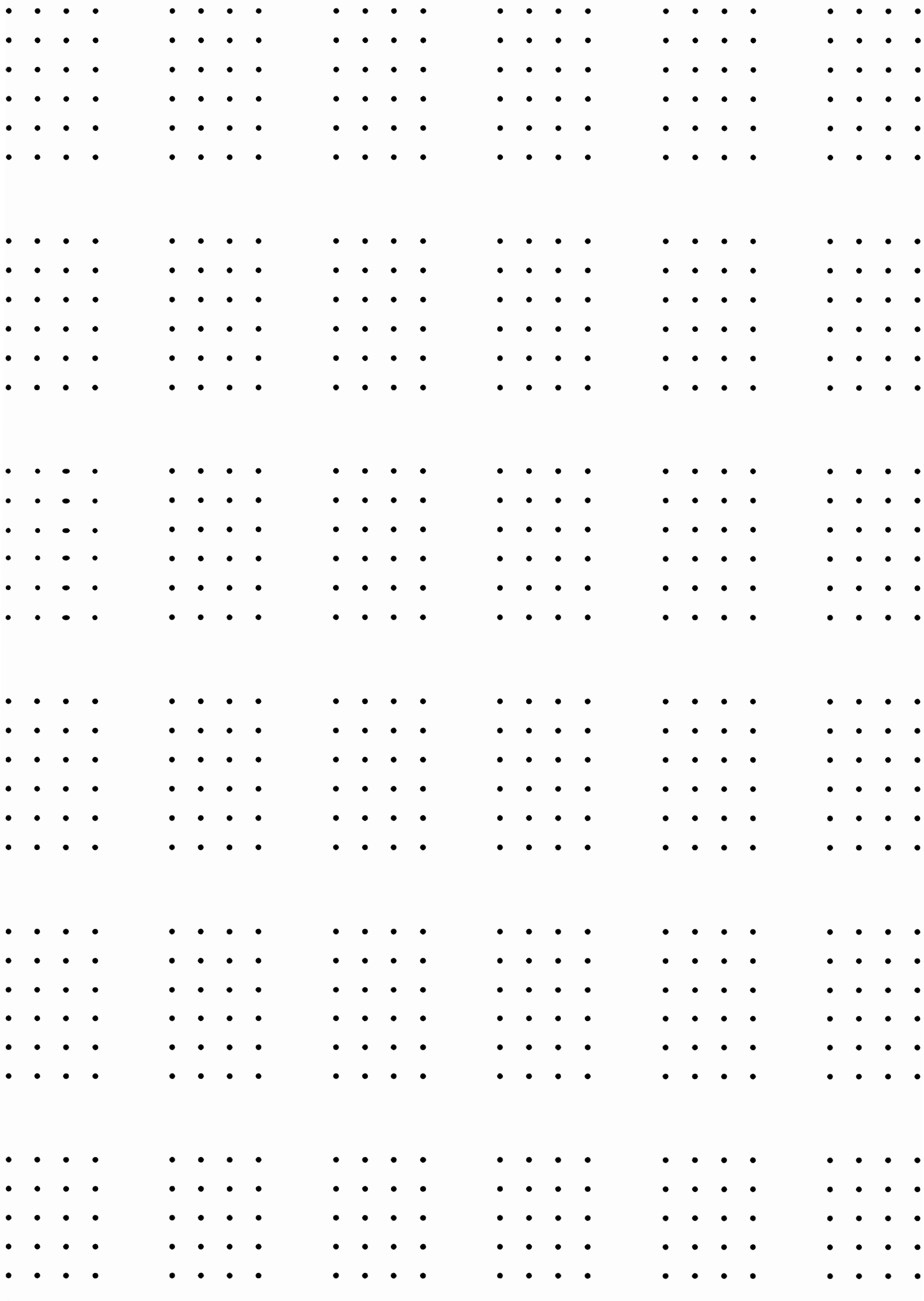
Power Question: ARML (4, 5) Polygons



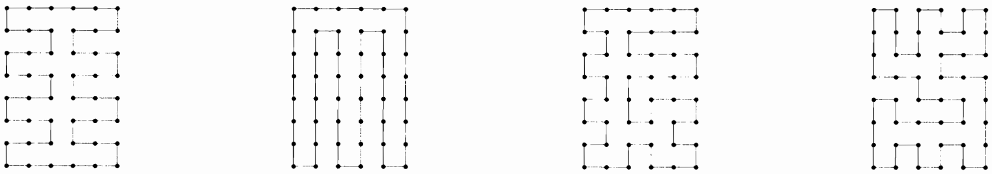
Power Question: ARML (4, 6) Polygons



Power Question: ARML (4, 6) Polygons--Continued

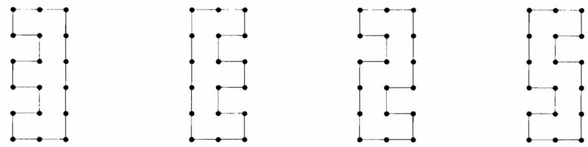


1. a) Shown below are 4 distinct ARML (6, 8) polygons. There are other possibilities.

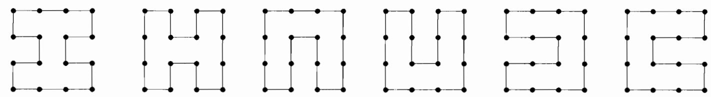


b) The area of each polygon is 23 and the perimeter of each is 48.

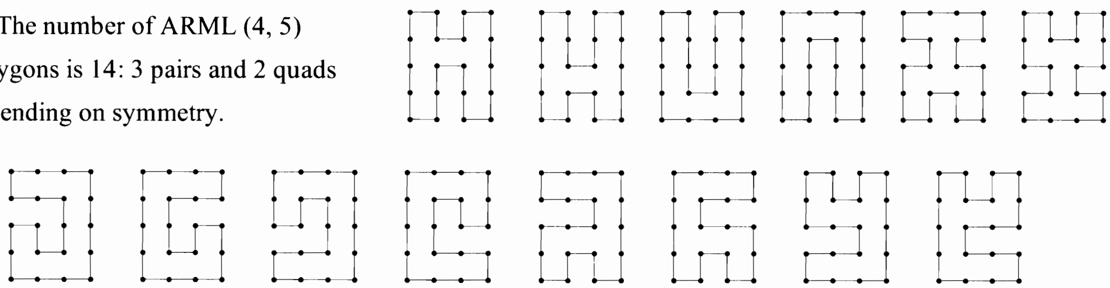
2. a) The number of ARML (3, 6) polygons is 4 .



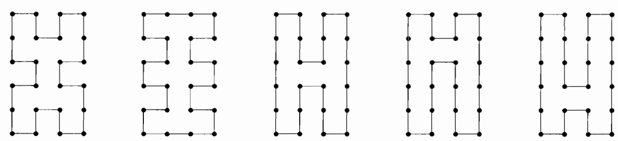
b) The number of ARML (4, 4) polygons is 6.

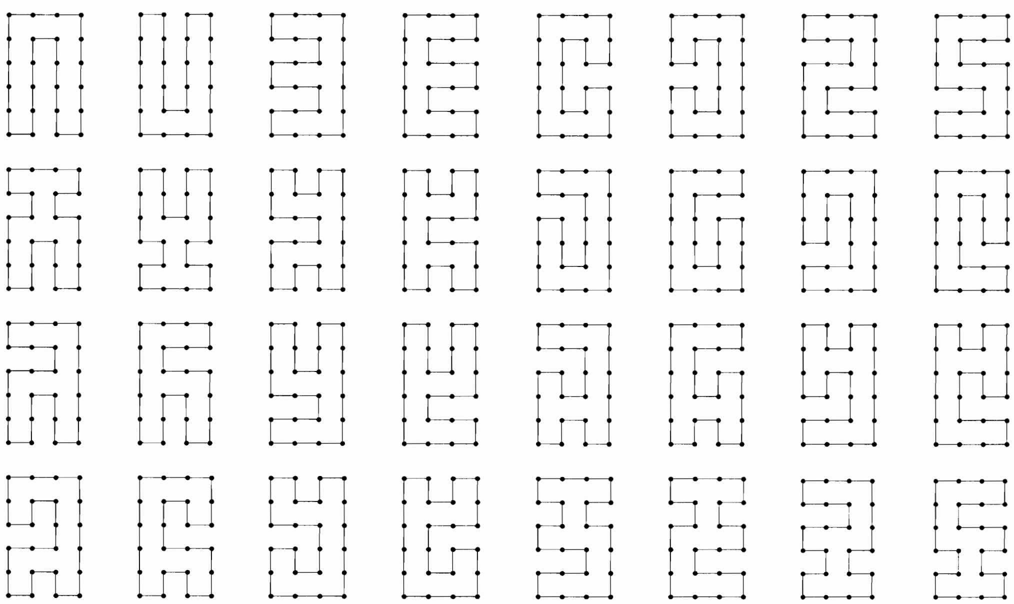


c) The number of ARML (4, 5) polygons is 14: 3 pairs and 2 quads depending on symmetry.



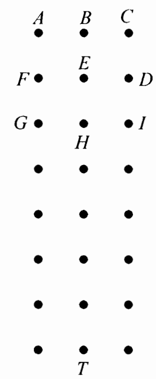
d) The number of ARML (4, 6) polygons is 37. These are grouped: 3 singles, 7 pairs, 5 quads, depending on symmetry.

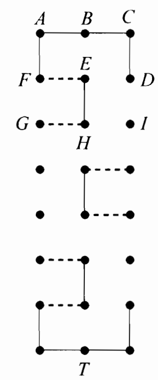




3. a) Rotate the array 90°. Starting from the corners, the path must look like the diagram at the right. The only choice is to continue to move horizontally, otherwise, some point will have 3 segments joining it to other points. Thus, $F(2, n) = 1$.

b) Given the left-hand array, point B on the central axis cannot connect with E since A and C would then lie off the polygon. Thus, the polygon must look like the right-hand diagram. Since n is even, there are $n - 2$ points in the central axis other than B and T which could be connected by a segment, giving $\frac{n-2}{2}$ segments. A segment $\overline{EH}$ can either be paired with an indent from D, namely $DEHI$, or an indent from F, namely $FEHG$. Thus, for each of the $\frac{n-2}{2}$ segments there are two choices. The total number of choices and hence, the total number of polygonal paths is $2^{(n-2)/2}$, making $F(3, n) = 2^{(n-2)/2}$.





4. We have $F(4, 0) = 0$, $F(4, 1) = 0$, $F(4, 2) = 1$, $F(4, 3) = 2$, $F(4, 4) = 6$, $F(4, 5) = 14$, and $F(4, 6) = 37$. Thus, for $n \geq 5$ we have the following formula which is consistent with these results: $F(4, n) = 2F(4, n-1) + 2F(4, n-2) - 2F(4, n-3) + F(4, n-4)$.

5. a) There are $m \cdot n$ lattice points. Since the segments can be put into a 1–1 correspondence with the lattice points by associating each segment with its starting point, the perimeter is mn.

b) As the polygonal path is traversed, every segment on which one moves to the right must be matched by a segment on which one moves to the left since the path finishes at the starting point. Similarly for up and down movements. Thus, the perimeter is even. But the perimeter equals $m \cdot n$, and if $m \cdot n$ is even, then at least one of m and n is even. Thus, if both are odd, there is no polygon.

<u>Note</u>: Since #6 is independent of #5b we could also prove #5b using #6. Since the ARML polygon consists of joined unit squares, its area is an integer. Thus, for the area, $\frac{mn}{2} - 1$, to be an integer, mn must be divisible by 2, making m, n, or both even, but not both odd.

6. Pick's Theorem states that the area of a simple closed polygon whose vertices are lattice points equals $q + \frac{p}{2} - 1$ where q is the number of lattice points in the interior of the polygon and p is the number of lattice points on the boundary. In the case of ARML polygons, there are no interior points since every point lies on the path. Thus $q = 0$ and $p = mn$. The area is, therefore, $\frac{mn}{2} - 1$.

Induction proof: it is easier to prove the result in the more general case rather than specifically for ARML (m, n) polygons. Consider any finite simple closed polygon of squares whose boundary consists of unit line segments that are horizontal or vertical, containing no vertices in its interior. Let its area be K, its perimeter be p and the number of vertices on the boundary be v. We will prove that $K = \frac{p}{2} - 1 = \frac{v}{2} - 1$ by induction on K. Let $K = 1$. The polygon is a square of side 1; clearly $p = v = 4$ and $K = \frac{4}{2} - 1$. Assume the result is true for $K = t$ and prove it for $K = t + 1$. Consider any such polygon with area $t + 1$. Then this polygon contains a square with three edges in common with the boundary (see justification below). Remove this square by erasing three edges and adding the fourth. Then, the area has been decreased by 1, the perimeter by 2 and the number of vertices by 2 and by induction, the result holds.

Justification: pick any square contained in the polygon. If it has 3 edges in common with the boundary, then we are done. Otherwise, move to one of the adjacent squares. If the new square has 3 edges on the boundary then we're done. Otherwise move to one of the adjacent squares, but not the one you just arrived from, and continue the process. Note that you can never return to a square you have previously visited (or else you will have traveled around a loop in your polygon which must contain a vertex, contradicting the fact that the polygon is simple). Then the process must end at some point and the only way it can end is by finding a square contained by the polygon with 3 edges in common with the boundary.

7. Let the area of an ARML (m, n) polygon be defined as $K(m, n)$. Then $K(4, y) - K(3, x) = d$ and $K(5, z) - K(4, y) = d$. From #6 we obtain $\left(\frac{4y}{2} - 1\right) - \left(\frac{3x}{2} - 1\right) = d$ and $\left(\frac{5z}{2} - 1\right) - \left(\frac{4y}{2} - 1\right) = d$ giving $4y - 3x = 2d$ and $5z - 4y = 2d$. Thus, $4y - 3x = 5z - 4y$, giving $3x - 8y + 5z = 0$. We need to choose distinct x, y, and z such that the areas are in an increasing arithmetic progression. Not all solutions will work. For example, from #5b neither x nor z can be odd. Additionally, while $x = 60, y = 30$, and $z = 12$ solves the Diophantine equation and has a sum greater than 100, that solution produces areas in a decreasing arithmetic sequence: 89, 59, and 29. But $x = 30, y = 40$, and $z = 46$ works, giving areas of 44, 79, and 114. There are infinitely many solutions.

8. Consider $K(3, 2r) = 20w$, $K(4, y) = 21w$, and $K(5, 2s) = 29w$. Then $3r - 1 = 20w$, $2y - 1 = 21w$, and $5s - 1 = 29w$. Clearly, w must be odd, otherwise y would be fractional. Let $w = 2v + 1$ for $v = 0, 1, 2, \ldots$ Then $3r - 1 = 40v + 20$, $2y - 1 = 42v + 21$, and $5s - 1 = 58v + 29$, giving $r = \frac{40v}{3} + 7$, $y = 21v + 11$, and $s = \frac{58v}{5} + 6$. Clearly, v must be a multiple of 15, so letting $v = 15t$ we obtain $r = 7 + 200t$, $y = 11 + 315t$, and $s = 6 + 174t$, giving ARML polygons $(3, 14 + 400t)$, $(4, 11 + 315t)$, and $(5, 12 + 348t)$. Letting $t = 1$, we obtain the following: $(3,414),(4,326),(5,360)$.

9. a) Setting $\frac{(m+7)(n+7)}{2} - 1 = 2\left(\frac{mn}{2} - 1\right)$ yields $n = \frac{51 + 7m}{m - 7} = 7 + \frac{100}{m - 7}$. Thus, $m - 7$ must divide $100 \rightarrow m - 7 \in \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$. Solving for m and n we obtain the following: (8, 107), (11, 32), (12, 27), (27, 12), (32, 11), (107, 8). We excluded (9, 57), (17, 17), and (57, 9) since in #5b we showed that at least one of m or n had to be even.

b) If $K(m,m) = \frac{1}{2}K(n,n)$, then $\frac{m^2}{2} - 1 = \frac{1}{2}\left(\frac{n^2}{2} - 1\right) \rightarrow 2(m^2 - 1) = n^2$. Thus, n^2 is even so let $n = 2k$. Then $2(m^2 - 1) = 4k^2 \rightarrow m^2 = 2k^2 + 1$, making m an odd number. But by #5b there are no ARML polygons in which both dimensions are odd, so there are no square arrays in which an ARML polygon can be half the area of another.

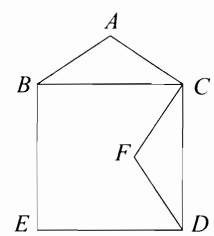
10. Setting $\frac{(m+k)(n+k)}{2} - 1 = 2\left(\frac{mn}{2} - 1\right)$ gives $n = \frac{k^2 + 2 + km}{m - k} = k + \frac{2(k^2 + 1)}{m - k}$. Let the prime factorization of $k^2 + 1$ be $2^{a_2}3^{a_3}5^{a_5}\ldots$. Then the prime factorization of $2(k^2 + 1)$ is $2^{a_2+1}3^{a_3}5^{a_5}\ldots$. Let $d = m - k$ be a divisor of $2(k^2 + 1)$ and note that any value of d for which either m or n or both are <u>even</u> results in an ARML polygon. There are several cases:

i) <u>k is even</u>. Then we claim that all factors d will give an ARML polygon. First, if d is even, then since $m = k + d$, m is even. If d is odd, then $\frac{2(k^2 + 1)}{d}$ is even, and since n then equals an even plus an even, n is even. Therefore, the number N of such polygons is $N = (a_2 + 2)(a_3 + 1)(a_5 + 1)\cdots$.

ii) <u>k is odd</u>. If d is odd, then $m = k + d$ is even. Thus, any factor d of the form $d = 3^{a_3}5^{a_5}7^{a_7}\ldots$ gives an ARML polygon. If d is even, then m is odd. In order for n to be even, $\frac{2(k^2 + 1)}{d}$ must be odd, implying that 2^{a_2+1} divides d. Thus, if k is odd, any factor d of the form $d = 2^{a_2+1}3^{a_3}5^{a_5}\ldots$ gives an ARML polygon. In counting the number of ARML polygons in this case, there are two choices for factors of 2, either no 2's or all 2's. Therefore, the number N of such polygons is $N = 2(a_3 + 1)(a_5 + 1)\ldots$.

I–1. For $1 < x < y$, let $S = \{1, x, y, x + y\}$. Compute the absolute value of the difference between the mean and the median of S.

I–2. $BCDE$ is a square, $\Delta ABC \cong \Delta FCD$ with $m\angle A = 120°$ and $AB = AC$. If $AF = 2000$, compute the area of square $BCDE$.



I–3. For x, y, z, and $w \geq 0$, compute the smallest value of x satisfying the following system:

$$\begin{aligned} y &= x - 2001 \\ z &= 2y - 2001 \\ w &= 3z - 2001 \end{aligned}$$

I–4. If $b = 2000$, compute the numerical value of the infinite sum:

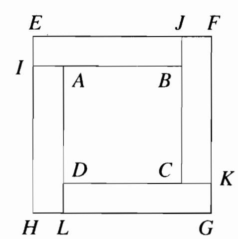
$$(\log_b 2)^0(\log_b 5^{4^0}) + (\log_b 2)^1(\log_b 5^{4^1}) + (\log_b 2)^2(\log_b 5^{4^2}) + (\log_b 2)^3(\log_b 5^{4^3}) + \ldots$$

I–5. If, from left to right, the last seven digits of $n!$ are 8000000, compute the value of n.

I–6. Let S be the sphere whose equation is $x^2 + y^2 + z^2 = 25$. Points $P(3,4,0)$ and $Q(3,-4,0)$ lie on S. Compute the number of planes containing P and Q that intersect S in a circle whose area is an integer.

I–7. The measure of the vertex angle of isosceles triangle ABC is θ and the sides of the triangle are $\sin\theta$, $\sqrt{\sin\theta}$, and $\sqrt{\sin\theta}$. Compute the area of ΔABC.

I–8. Square $ABCD$ is surrounded by four congruent rectangles. Let
S = perimeter of $ABCD$ + perimeter of $EFGH + AI + BJ + CK + DL$.
Let T = area of $EFGH$. Let m be the greatest lower bound and n be the least upper bound of all values of T for which $\dfrac{S}{T} = 12$. Compute the ordered pair (m, n).



1. $\frac{1}{4}$

2. 6,000,000

3. 3335

4. $\frac{1}{3}$

5. 27

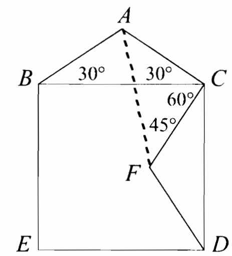
6. 56

7. $\frac{8}{25}$

8. $\left(\frac{1}{4}, \frac{4}{9}\right)$

I–1. The mean is $\dfrac{1 + x + y + (x + y)}{4} = \dfrac{1}{4} + \dfrac{2(x + y)}{4}$, the median is $\dfrac{x + y}{2}$ and the difference is $\boxed{\dfrac{1}{4}}$.

I–2. Since $AB = AC$, $m\angle ACB = 30°$. Since $\Delta ABC \cong \Delta FDC$, then $m\angle FCD = 30° \to m\angle FCB = 60°$. Thus, $m\angle ACF = 90°$ and since $AC = FC$, ΔACF is a 45-45-90 right triangle. Let $AF = x \to$ $AC = \dfrac{x}{\sqrt{2}}$. Use the Law of Cosines on ΔABC or drop an altitude from A to use 30-60-90 right triangles to obtain $BC = \dfrac{x\sqrt{3}}{\sqrt{2}}$. Thus, the area of $BCDE = \dfrac{3x^2}{2} = \dfrac{3 \cdot 2000^2}{2} = \boxed{6{,}000{,}000}$.



I–3. Since $w \ge 0$, $3z - 2001 \ge 0 \to z \ge 667$. Since $z = 2y - 2001$, $2y - 2001 \ge 667 \to y \ge 1334$. Since $y = x - 2001$, $x - 2001 \ge 1334 \to x \ge 3335$. The smallest value of x is $\boxed{3335}$.

<u>Alternate solution</u>: multiply the first equation by 6 and the second by 3:

$$\begin{aligned} 6y &= 6x - 12006 \\ 3z &= 6y - 6003 \\ w &= 3z - 2001 \end{aligned}$$

Adding yields $6y + 3z + w = 6x + 6y + 3z - 20010 \to w = 6x - 20010 \to x = \dfrac{w + 20010}{6}$.

Since $w \ge 0$, the minimum value for x is $\dfrac{20010}{6} = 3335$.

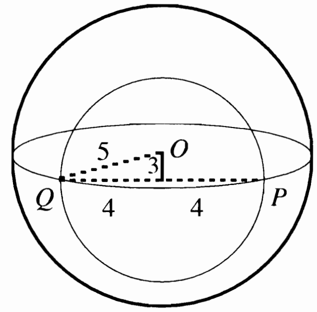
I–4. $1 \cdot \log_b 5 + (\log_b 2)(4)\log_b 5 + (\log_b 2)^2(16)\log_b 5 + (\log_b 2)^3(64)\log_b 5 + \cdots =$

$\log_b 5\left(1 + (4\log_b 2) + (4\log_b 2)^2 + (4\log_b 2)^3 + \cdots\right) = \log_b 5\left(\dfrac{1}{1 - 4\log_b 2}\right) =$

$\log_b 5\left(\dfrac{1}{\log_b b - \log_b 16}\right) = \dfrac{\log_b 5}{\log_b \frac{b}{16}} = \log_{b/16} 5$. Since $b = 2000$, we have $\log_{125} 5 = \boxed{\dfrac{1}{3}}$.

I–5. Since $n!$ ends in 6 zeros, $n!$ must have exactly 6 fives in its prime factorization, so we know that $25 \le n \le 29$. Consider the units digit of $\dfrac{n!}{10^6}$. For $n = 25$, a quick multiplication by all remaining factors yields a units digit of 4. For $n = 26$, the units digit is also 4, but for $n = 27$ it is 8. Thus, $n = \boxed{27}$.
Note that for $n = 28$, the units digit is 4 and for $n = 29$, it is 6.

I–6. The largest circle through P and Q is the horizontal circle containing the center of the sphere. It has a radius of 5. The smallest circle is the vertical circle with radius 4. Let A be the area of a circle formed by the intersection of a plane and sphere S. Then $16\pi \le A \le 25\pi$ giving $50.26 \le A \le 78.54$. If A is an integer, then $51 \le A \le 78 \rightarrow 28$ integral values for A. The plane can tilt from the horizontal up to the vertical or down to the vertical making a total of $\boxed{56}$ planes intersecting S in a circle of integral area.



I–7. By the Law of Cosines, $\sin^2\theta = \sin\theta + \sin\theta - 2\sin\theta\cos\theta$. Since $\sin\theta \ne 0$, divide by $\sin\theta$ to obtain

$\sin\theta = 2 - 2\cos\theta \rightarrow \sin^2\theta = (2-2\cos\theta)^2 \rightarrow 1-\cos^2\theta = 4\cos^2\theta - 8\cos\theta + 4 \rightarrow$

$5\cos^2\theta - 8\cos\theta + 3 = 0$. Then $\cos\theta = 1$ or $\frac{3}{5}$. Reject 1 since it makes $\theta = 0$. Thus, $\sin\theta = \frac{4}{5}$ and the

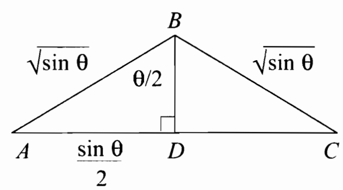
area of $\Delta ABC = \frac{1}{2}\sqrt{\sin\theta}\cdot\sqrt{\sin\theta}\cdot\sin\theta = \frac{1}{2}\cdot\frac{4}{5}\cdot\frac{4}{5} = \boxed{\frac{8}{25}}$.

Alternate solution: Since $\sin\frac{\theta}{2} = \frac{AD}{AB} = \frac{AC}{2AB} = \frac{\sin\theta}{2\sqrt{\sin\theta}}$, then

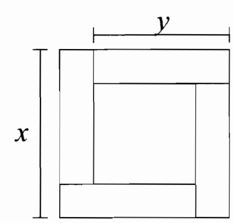
$2\sin\frac{\theta}{2} = \sqrt{\sin\theta} \rightarrow 4\sin^2\frac{\theta}{2} = \sin\theta$. Since $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$,

divide by $2\sin\frac{\theta}{2}$ to obtain $2\sin\frac{\theta}{2} = \cos\frac{\theta}{2} \rightarrow \tan\frac{\theta}{2} = \frac{1}{2}$. Since

$\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$, then $\sin\theta = \frac{4}{5}$. Continue as above.



I–8. Let the side of the outer square be x and the length of each rectangle be y. Then $T = x^2$ and $S = 4x + 4y$. Hence, $\frac{4x+4y}{x^2} = 12 \rightarrow y = 3x^2 - x$. There are two degenerate cases, one where the interior square disappears, suggested by Fig. 1 and the other where the interior square fills the outer square, suggested by Fig. 2.



In the former, $y = \frac{x}{2}$ and in the latter, $y = x$.

Thus, $\frac{x}{2} < y < x \rightarrow \frac{x}{2} < 3x^2 - x < x \rightarrow \frac{1}{2} < x < \frac{2}{3}$.

Thus, $\frac{1}{4} < T < \frac{4}{9}$. Ans: $\boxed{\left(\frac{1}{4},\frac{4}{9}\right)}$.

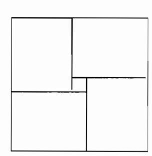


Fig. 1

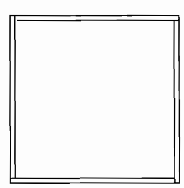


Fig. 2

ARML Relay #1 – 2000

R1–1. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Compute the largest integer n such that $\left\lfloor \sqrt[n]{2000} \right\rfloor > 1$.

R1–2. Let T = TNYWR. Compute the larger solution to $\left(\log_T x\right)^2 = \log_T x^2$.

R1–3. Let T = TNYWR. Let $n \in \left\{1, 2, 3, \ldots, T^2\right\}$. Compute the number of values of n for which the product $(n^2 - 2n + 2)(n^2 + 2n + 2)$ is divisible by 5.

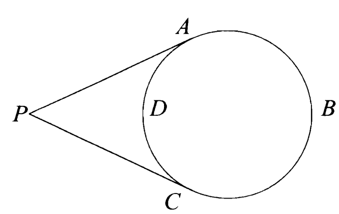
ARML Relay #2 – 2000

R2–1. For $0 < x < 2\pi$, compute the absolute value of the difference between the solutions to the equation: $\sec x = 1 + \cos x + \cos^2 x + \cos^3 x + \ldots$

R2–2. Let T = TNYWR and set $K = \frac{4\pi}{T}$. If $(a - bi)^K (a + bi)^K = 512$, compute the value of $a^2 + b^2$.

R2–3. Let T = TNYWR. Arcs are measured in degrees.

If $\frac{m\overparen{ABC}}{m\overparen{ADC}} = T + 1$, compute $\frac{m\overparen{ADC}}{m\angle P}$.



Relay #1:

R1–1. 10

R1–2. 100

R1–3. 8000

Relay #2:

R2–1. $\frac{4\pi}{3}$

R2–2. 8

R2–3. $\frac{1}{4}$

Solutions to ARML Relay # 1 – 2000

R1–1. Since we want $\sqrt[n]{2000} \geq 2$, then $2000 \geq 2^n$. Since $2^{10} < 2000 < 2^{11}$, $n = \boxed{10}$.

R1–2. $\left(\log_T x\right)^2 - 2\log_T x = 0 \rightarrow \left(\log_T x\right)\left(\log_T x - 2\right) = 0 \rightarrow x = T^0$ or T^2. Since $T = 10$, the larger value of $x = 10^2 = \boxed{100}$.

R1–3. Since $T = 100$, then $n \in \{1, 2, \ldots, 10{,}000\}$. Since $(n^2 - 2n + 2)(n^2 + 2n + 2) = n^4 + 4$, then n^4 must end in 1 or 6 $\rightarrow$ n must end in 1, 2, 3, 4, 6, 7, 8, 9 $\rightarrow$ n doesn't end in 0 or 5. There are $\dfrac{10{,}000}{5} = 2000$ numbers for which n^4 does not end in 1 or 6, so there are $\boxed{8000}$ which do.

Or consider:

n (mod 5)	$n^2 - 2n + 2$ (mod 5)	$n^2 + 2n + 2$ (mod 5)
0	2	2
1	1	0
2	2	0
3	0	2
4	0	1

The product of the two numbers in the first row is not divisible by 5; the products in the other rows are. Since $T^2 = 10{,}000$ and that is a multiple of 5, four–fifths of the numbers in the set will give a result divisible by 5.

Solutions to ARML Relay # 2 – 2000

R2–1. $\sec x = \dfrac{1}{1 - \cos x} \rightarrow \sec x - 1 = 1 \rightarrow \sec x = 2 \rightarrow \cos x = \dfrac{1}{2} \rightarrow x = \dfrac{5\pi}{3}$ or $\dfrac{\pi}{3}$.
The desired difference is $\boxed{\dfrac{4\pi}{3}}$.

R2–2. $K = 3$. Since $(a - bi)^K (a + bi)^K = \left((a - bi)(a + bi)\right)^K = (a^2 + b^2)^K$, then $a^2 + b^2 = 512^{1/K}$.

Thus, $a^2 + b^2 = 512^{1/3} = \boxed{8}$.

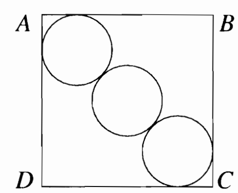
R2–3. Let the measure of arc $ADC = x$, the measure of arc $ABC = 360° - x$ and $m\angle P = 180° - x$. Then,
$\dfrac{360° - x}{x} = T + 1 \rightarrow \dfrac{360°}{T + 2} = x$. Also $180° - x = 180° - \dfrac{360°}{T + 2} = \dfrac{180T}{T + 2}$. Thus,
$\dfrac{x}{180° - x} = \dfrac{2}{T}$. Since $T = 8$, the ratio is $\boxed{\dfrac{1}{4}}$.

Note: Pass answers from position 1 to 8 and from position 15 to 8.

1. The letters A, R, M, and L represent even digits. A, R, M, and L are not necessarily distinct and A is not equal to zero or to L. Compute the positive integral value of b such that $\sqrt{ARML_b} = 2^5$.

2. Let T = TNYWR, but T is irrelevant here. $P(x) = x^3 + 2x^2 - (4K + 5)x - 6$ has three distinct zeros. If two of these zeros are the same as those of $f(x) = x^2 + 5x + K$, compute K.

3. Let T = TNYWR. The centers of three congruent circles lie on the diagonal of square $ABCD$. The two end circles are tangent to the square and to the middle circle. If the radius of each circle equals $\sqrt{T} - 1$, compute the area of $ABCD$.



4. Let T = TNYWR. Compute the number of integers which lie in both of the intervals $[\, T - 9, 2T + 10\,]$ and $[\, T - 4, 2T + 13\,]$.

5. Let T = TNYWR. Suppose T elements (not necessarily distinct) are selected from the set $\{0, 1, 2, 3, 4, 5\}$. Compute the number of distinct sums of these T elements.

6. Let T = TNYWR. Let $K = \dfrac{T}{-8}$. Compute the value of y solving:

$$\begin{aligned} 6x + 2y &= K \\ -7x + 3y &= 30 \end{aligned}$$

7. Let T = TNYWR. Compute the number of distinct T-letter words that can be formed using the letters from the word ARRANGE.

8. You will receive two numbers. Let M be the larger of the two and let m be the smaller. Compute the area of the set of points (x, y) satisfying: $m \le \left(|x| - 2000\right)^2 + \left(|y| - 2000\right)^2 \le M$.

15. The letters A, R, M, and L represent even digits. A, R, M, and L are not necessarily distinct and A is not equal to zero or to L. Compute the positive integral value of b such that $\sqrt{ARML_b} = 2^5$.

14. Let T = TNYWR, but T is irrelevant here. $P(x) = x^3 + 2x^2 - (4K + 5)x - 6$ has three distinct zeros. If two of these zeros are the same as those of $f(x) = x^2 + 5x + K$, compute K.

13. Let T = TNYWR. In ΔABC, $AB = AC = 10$ and $BC = 2T$. Point P is chosen at random on $\overline{BC}$, D is chosen on $\overline{AB}$ and E on $\overline{AC}$ so that $\overline{PD}$ and $\overline{PE}$ are parallel to $\overline{AC}$ and $\overline{AB}$ respectively. Compute the perimeter of $PDAE$.

12. Let T = TNYWR. The number 2000^T has x terminal zeros when expanded. Let the rightmost non-zero digit be y. Compute $x + y$.

11. Let T = TNYWR and set $K = \frac{-T}{22}$. Points $A(1, 2)$, $B(5, K)$, and $C(K, 7)$ are collinear. Compute the y-intercept of $\overleftrightarrow{AB}$.

10. Let T = TNYWR and set $K = \frac{12T}{13}$. In parallelogram $ABCD$, if $\cos B = \frac{K^2 - 1}{K^2 + 1}$, compute $\sin A$.

9. Let T = TNYWR and set $K = 90T$. $ABCD$ is an isosceles trapezoid with $\overline{AB}$ parallel to $\overline{CD}$. If $BC = 5(AB)$, $CD = 7(AB)$, and the perimeter of $ABCD = K$, compute the height of $ABCD$.

8. You will receive two numbers. Let M be the larger of the two and let m be the smaller. Compute the area of the set of points (x, y) satisfying: $m \le \left(|x| - 2000\right)^2 + \left(|y| - 2000\right)^2 \le M$.

1. 8

2. 2

3. 4

4. 19

5. 96

6. 3

7. 84

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15. 8

14. 2

13. 20

12. 66

11. $\frac{13}{4}$

10. $\frac{3}{5}$

9. 12

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8. 288π

1. $Ab^3 + Rb^2 + Mb + L = 2^{10} = 1024$. If $b = 9$ or 10, then $A = 1$, but A is to be even. Try $b = 8$. Then $512A + 64R + 8M + L = 1024$ and clearly $A = 2$, $R = M = L = 0$ works. If $b = 7$, then $2662_7 = 1024$, but $A = L$. If $b = 6$, then $4424_6 = 1024$, but $A = L$. Thus, $b = \boxed{8}$, giving $\sqrt{2000_8} = 2^5$, fitting for ARML's 25th anniversary.

2. $\dfrac{x^3 + 2x^2 - (4K + 5)x - 6}{x^2 + 5x + K} = (x - 3)$ with a remainder of $15x + 3K - (5K + 5)x - 6$. The remainder is zero if $K = \boxed{2}$.

3. Let r be the radius of the circles. Using 45-45-90 right triangles, $AC = 4r + 2r\sqrt{2}$. Since $T = 2$, $r = \sqrt{2} - 1$, making $AC = 2\sqrt{2}(\sqrt{2} + 1)(\sqrt{2} - 1) = 2\sqrt{2}$. Thus, $AB = 2$ and the area of $ABCD = \boxed{4}$.

4. Common integers lie in $[T - 4, 2T + 10] \to (2T + 10) - (T - 4) + 1 = T + 15$ integers. Since $T = 4$, then there are $\boxed{19}$ integers.

5. Using T 0's the sum is 0. By replacing each integer n by $n + 1$, we can obtain all sums from $T \cdot 0$ to $T \cdot 5$. Thus, there are $5T + 1$ sums. Since $T = 19$, $5T + 1 = \boxed{96}$.

6. Multiply the top equation by 7 and the bottom by 6 and add, canceling the x's and obtaining $32y = 180 + 7K$. Since $K = \dfrac{96}{-8} = -12$, $y = \dfrac{96}{32} = \boxed{3}$.

7. If $T = 1$, there are 5 distinct words. If $T = 2$, there are two words using RR or AA and $5 \cdot 4 = 20$ using ARNGE. Total: 22. If $T = 3$, then with two R's and another letter we obtain $\dfrac{3!}{2!} \cdot 4 = 12$ words. Similarly for two A's and another letter. Using ARNGE there are $5 \cdot 4 \cdot 3 = 60$ words. Since $T = 3$, there are $60 + 12 + 12 = \boxed{84}$ words.

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15. See #1 above. Ans: $\boxed{8}$.

14. See #2 above. Ans: $\boxed{2}$.

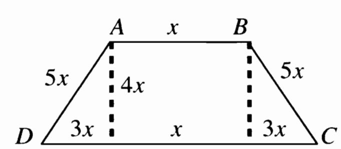
13. T is irrelevant; the result is invariant. Since ΔPEC is isosceles, $PE = EC$ so $AE + EP = AE + EC = AC$. Similarly, $AD + DP = AD + DB = AB$. Perimeter of $PDAE = AB + AC = \boxed{20}$.

12. $\left(2 \cdot 10^3\right)^T = 2^T \cdot 10^{3T}$. Thus, $x = 3T$ and for $T = 20$, $x = 60$. Since 2^{20} ends in 6, $x + y = \boxed{66}$.

11. $m\overline{AB} = m\overline{AC} \rightarrow \dfrac{K-2}{4} = \dfrac{5}{K-1} \rightarrow K^2 - 3K - 18 = 0 \rightarrow K = -3$ or 6. If $K = 6$, the line is $y = x + 1$ and the intercept is 1. If $K = -3$, the line is $y = \dfrac{-5x}{4} + \dfrac{13}{4}$ and the y-intercept is $\dfrac{13}{4}$. You could pass each back in turn. Since $K = -3$, the answer is $\boxed{\dfrac{13}{4}}$.

10. Since adjacent angles of a parallelogram are supplementary, $\sin A = \sin B$. Thus, $\sin^2 B = 1 - \dfrac{(K^2-1)^2}{(K^2+1)^2} = \dfrac{4K^2}{\left(K^2+1\right)^2}$. Thus, $\sin B = \dfrac{|2K|}{\left(K^2+1\right)}$. Since $K = \dfrac{12}{13} \cdot \dfrac{13}{4} = 3$, $\sin B = \boxed{\dfrac{3}{5}}$.

9. From the diagram, we clearly have 3-4-5 right triangles, so the height is $4x$. The perimeter is $18x$, so $18x = 90 \cdot \dfrac{3}{5} = 54 \rightarrow x = 3 \rightarrow$ height $= \boxed{12}$.



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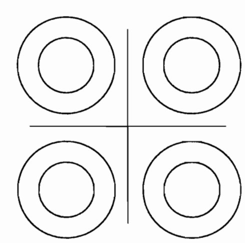
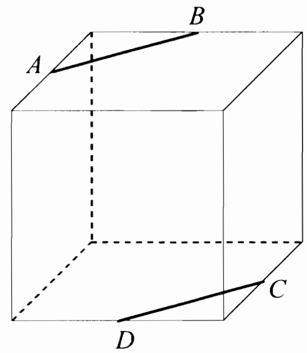
8. By replacing $|x|$ and $|y|$ with x or $-x$, y or $-y$, we obtain a graph consisting of 4 rings formed by 4 sets of concentric circles. As long as M is small compared to the distance to the axes, the rings do not overlap. That is the case here. The radii of the two circles are $\sqrt{m}$ and $\sqrt{M}$, and the area of each ring is $\pi M - \pi m$. Since $M = 84$ and $m = 12$, the total area equals $4\pi(84 - 12) = \boxed{288\pi}$.



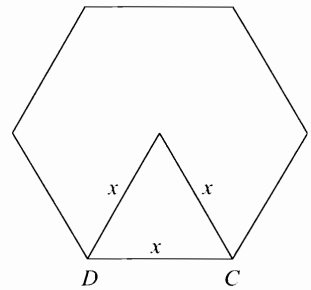
1. Points A, B, C, and D are the midpoints of the edges of the cube as shown. A plane passing through $\overline{AB}$ and $\overline{CD}$ intersects the cube in a region whose area is $\sqrt{6}$. Compute the surface area of the cube.



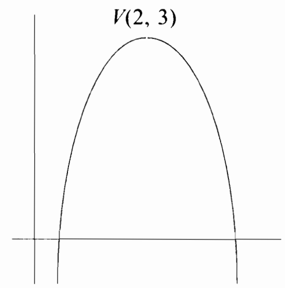
2. Let $V(2, 3)$ be the vertex of a parabola whose equation is of the form $y = ax^2 + bx + c$. If the graph of the parabola has two positive roots, compute the greatest integral value of k such that $a = \dfrac{k}{2000}$.

3. Perpendicular chords $\overline{AB}$ and $\overline{CD}$ of circle O intersect at P. If the radius of O is 10 and $OP = 4$, compute $AB^2 + CD^2$.

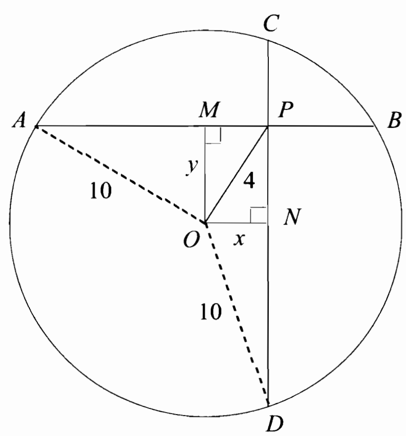
1. A plane passing through $\overline{AB}$ and $\overline{DC}$ intersects the cube in regular hexagon of side $\overline{DC}$. Let $DC = x$. Then the area of the hexagon equals $6 \cdot \dfrac{x^2\sqrt{3}}{4} = \sqrt{6} \rightarrow x = \dfrac{\sqrt[4]{8}}{\sqrt{3}}$. Thus, $DC = \dfrac{\sqrt[4]{8}}{\sqrt{3}}$. The length of the edge of the cube equals $\dfrac{DC}{\sqrt{2}} \cdot 2 = DC\sqrt{2}$. Thus, the surface area of the cube equals $6\left(DC \cdot \sqrt{2}\right)^2 = \dfrac{12\sqrt{8}}{3} = \boxed{8\sqrt{2}}$.



2. Since $\dfrac{-b}{2a} = 2$, then $b = -4a$. Since $3 = a \cdot 2^2 - (4a)2 + c$, then $c = 4a + 3$. For there to be two positive roots, c must be negative. Thus, $4a + 3 < 0 \rightarrow a < -\dfrac{3}{4}$. Hence, $\dfrac{k}{2000} < \dfrac{-3}{4} \rightarrow k < -1500$. The largest integral value of k is $\boxed{-1501}$.



3. Let $AM = \sqrt{100 - y^2}$ making $AB = 2\sqrt{100 - y^2}$. Let $ND = \sqrt{100 - x^2}$ making $CD = 2\sqrt{100 - x^2}$. Then $AB^2 + CD^2 = 4\left(100 - y^2\right) + 4\left(100 - x^2\right) = 800 - 4\left(x^2 + y^2\right)$. Since $x^2 + y^2 = 16$, then $AB^2 + CD^2 = 800 - 64 = \boxed{736}$.



ARML

2001

THE 26th ANNUAL MEET

For the 26th competition, ARML opened up a new site. Since the site in Las Vegas was proving to be too small, the Executive Board spent the year looking for a new site and finally found the perfect one at San Jose State University in California. Mark Saul retired as president of ARML to take up his new duties as a director of the NCTM and Tim Sanders, director of the Great Plains Mathematics League and corresponding secretary for ARML, took over as president.

This year proved to be ARML's largest yet. There were 25 teams in Division A, 76 teams in Division B and 7 guest teams, 6 from Taiwan and, for the first time, a team from the Philippines. The total of 108 teams meant that well over 1600 students as well as countless teachers and coaches participated in the contest. In Division A, San Francisco Bay A edged out Massachusetts A. They were tied after the team round and the Power Question and the relays equalized each other, but with a narrow victory in the individual round, SFBA gained a 6 point victory. In Division B, Ontario B1 won easily after amassing an 18 point advantage in the individual round. ARML also introduced the ARML song contest in which teams competed to come up with the best song with mathematical content. It was great fun.

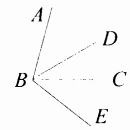
Tim Sanders received the Samuel Greitzer Distinguished Coach Award. Tim is currently the new president of ARML and has served on the Executive Board as a secretary. He has worked with the Kansas and Missouri teams.

John Benson of Chicago A received the Alfred Kalfus Founder's Award for his long time service to ARML. John has been a dedicated organizer of teams and a site coordinator at Iowa.

The following students received the Zachary Sobol Award for outstanding service to their team:

Brentan Alexander	Colorado
Andrew Dexter	Western Massachusetts
Jeff Huang	Chicago
Jennifer Philbrick	Maine

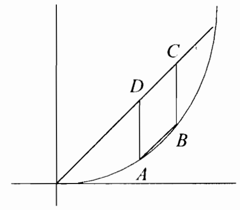
T–1. $\overline{AB} \perp \overline{BC}$, $\overline{BD} \perp \overline{BE}$, $m\angle ABD = 3x$, $m\angle DBC = x + y$, and $m\angle CBE = 2y$. Compute the value of y in degrees.



T–2. If each distinct letter in $\sqrt{ARML} = AL$ represents a distinct positive digit in base 10, compute the sum $A + R + M + L$.

T–3. The Luzors played y games, winning some and losing the rest. They then won 3 in a row and improved their winning percentage by exactly 10%. Compute the least value of y.

T–4. Rhombus $ABCD$ is inscribed in $y = x$ and $y = x^2$ with $\overline{CD}$ lying on $y = x$ and $\overline{AD}$ parallel to the y-axis. Compute the sum of the coordinates of A. (*Diagram not drawn to scale*).



T–5. Let $T = \{1, 2, 3, 4, \ldots, n\}$ and let $S(X)$ denote the sum of the elements in set X. Suppose T can be divided into three disjoint subsets A, B, and C whose union is T. If $S(A) : S(B) : S(C) = 3 : 4 : 6$, then compute the smallest possible value for the largest element in C.

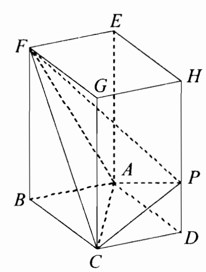
T–6. Compute the ratio of the volume of a sphere to the volume of the largest regular octahedron that will fit inside it.

T–7. In a square of side n, counting numbers are arranged in an inward spiral. For example, the diagram shows the result when $n = 5$. If $n = 27$, compute the sum of the elements in the diagonal from the top left to the bottom right.

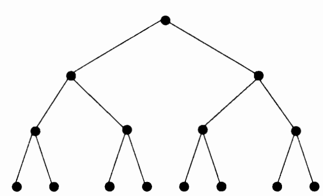
1	2	3	4	5
16	17	18	19	6
15	24	25	20	7
14	23	22	21	8
13	12	11	10	9

T–8. Compute the number of sets of three distinct elements that can be chosen from the set $\left\{2^1, 2^2, 2^3, \ldots, 2^{1999}, 2^{2000}\right\}$ such that the three elements form an increasing geometric progression.

T–9. Let $ABCDEFGH$ be a rectangular box such that $AB = AD = 20$ and $m\angle GAC = 45°$. Point P lies on $\overline{DH}$ such that plane PAC is parallel to $\overline{BH}$. Compute the volume of tetrahedron $FPCA$.



T–10. A tree is a connected graph with no circuits. A tree contains at least one node. To the right is a tree with 15 nodes. Define a subtree of a tree to be a subgraph that is also a tree, i.e., a nonempty subset of the nodes along with the edges joining them that together form a connected set. Compute the number of subtrees in the figure.



1. $\frac{270}{11}$ or $24.\overline{54}$

2. 18

3. 12

4. $2-\sqrt{2}$

5. 9

6. π or $\pi:1$

7. 12767

8. 999,000

9. $2000\sqrt{2}$

10. 750

T–1. $(3x)+(x+y)=90$ and $(x+y)+2y=90 \rightarrow 4x+y=90$ and $x+3y=90$. Solving $y = \boxed{\dfrac{270}{11} \text{ or } 24.\overline{54}}$.

T–2. Since $0 < L < 10$, $1000A + 100R + 10M + L < (10A+10)^2$. Drop off $10M+L$ and divide by 100 to obtain $10A + R < A^2 + 2A + 1$. Thus, $A^2 - 8A + 16 > R + 15 \rightarrow (A-4)^2 > R + 15$.

Since $R \geq 1$, $(A-4)^2 > 16$, making $A = 9$. Since the square of AL ends in L, then $L = 1, 5$, or 6, making $AL = 91, 95$, or 96. Now, $91^2 = 8281$ fails because A and M are not distinct and $95^2 = 9025$ fails because R is not positive. But $96^2 = 9216$ works, giving the sum $9+2+1+6 = \boxed{18}$.

T–3. Let x be the number of games won. Then $\left(\dfrac{x}{y}\right)\dfrac{11}{10} = \dfrac{x+3}{y+3} \rightarrow y = \dfrac{33x}{30-x}$. Let $z = 30 - x$. Then

$y = \dfrac{33(30-z)}{z} = \dfrac{990}{z} - 33 = \dfrac{2 \cdot 3^2 \cdot 5 \cdot 11}{z} - 33$. Since y is a positive integer, z must be a divisor of 990 such that the quotient is greater than 33. Of these 11 candidates for (z, y): $(1, 957)$, $(2, 462)$, $(3, 297)$, $(5, 165)$, $(6, 132)$, $(9, 77)$, $(10, 66)$, $(11, 57)$, $(15, 33)$, $(18, 22)$, and $(22, 12)$, $z = 22$ gives the minimum value of y as $\boxed{12}$. Thus, the team improved its winning percentage by exactly 10% as it went from $\dfrac{8}{12}$ to $\dfrac{11}{15}$.

<u>Note</u>: one coach thought that the problem was ambiguous in that an increase of 10% could mean adding 10% to the original winning percentage. If the problem is interpreted in that way, we have the following equation:

$\dfrac{100x}{y} + 10 = \dfrac{100(x+3)}{y+3} \rightarrow x = \dfrac{27y - y^2}{30}$. Interestingly, the smallest solutions to this equation are $x = 6$ and $y = 12$, so regardless of the interpretation, the least value for y is still 12.

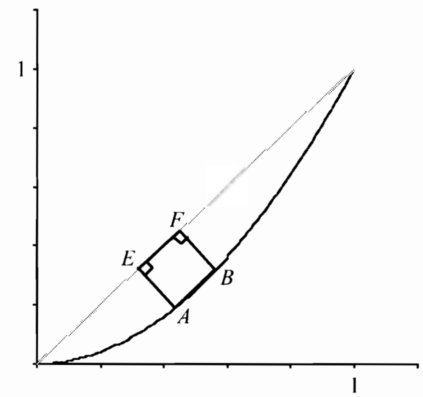
T–4. Starting with $D(a, a)$ and $C(b, b)$, we have $A(a, a^2)$ and $B(b, b^2)$. The slope of $\overline{AB} = 1$, so

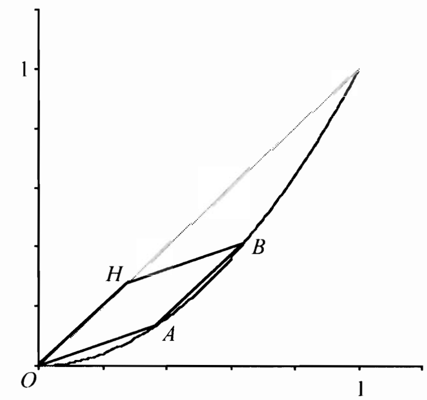
$\dfrac{b^2 - a^2}{b - a} = b + a = 1$. Since $AD = DC$, $a - a^2 = \sqrt{(b-a)^2 + (b-a)^2} = (b-a)\sqrt{2}$.

Since $b = 1 - a$, $a - a^2 = (1 - 2a)\sqrt{2}$. Solving the quadratic $a^2 - (2\sqrt{2}+1)a + \sqrt{2} = 0$ gives

$a = \dfrac{\left(2\sqrt{2}+1\right) \pm \sqrt{8 + 4\sqrt{2} + 1 - 4\sqrt{2}}}{2} = \dfrac{(2\sqrt{2}+1) \pm 3}{2}$. This gives $a = \sqrt{2} + 2$ which we reject since it is too large or $a = \sqrt{2} - 1 \approx 0.588$ which works, making $a + a^2 = \boxed{2 - \sqrt{2}}$.

Note: As originally stated, the problem did not specify that $\overline{AD}$ and $\overline{BC}$ were vertical. In the midst of grading the Team round, we suddenly realized that more than one rhombus was possible, as indicated in the diagrams below, and that, therefore, there would be a range of values for $a + a^2$.

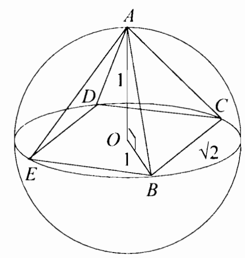




Sasha Schwarz found the range of $a + a^2$ in the following way: because $\overline{AB}$ is parallel to $y = x$, we have $A\left(a, a^2\right)$ and $B\left(1 - a, (1 - a)^2\right)$. The length of $\overline{AB}$ will range from AE, the perpendicular distance from A to $y = x$, to AO, the distance from A to the origin O. Using the formula for the distance from a point to a line, $AE = \dfrac{a - a^2}{\sqrt{2}}$ and clearly, $OA = \sqrt{a^2 + a^4}$. Given that $AB = \sqrt{2}(1 - 2x)$ we have

$\dfrac{a - a^2}{\sqrt{2}} \leq \sqrt{2}(1 - 2a) \leq a^2 + a^4$. Using a calculator, we find that $0.3633 \leq a \leq 0.4384$, and since $y = a + a^2$ is an increasing function on [0.3633, 0.4384], we have the following range of values for $a + a^2$: $0.495 \leq a + a^2 \leq 0.631$. We gave credit to any answer lying in that interval.

T–5. Let $S(A) = 3x$, $S(B) = 4x$, and $S(C) = 6x$. Now $S(A) + S(B) + S(C) = \dfrac{n(n+1)}{2}$. This must be divisible by 3 + 4 + 6 = 13 which is prime so either n or $n + 1$ is a multiple of 13. Thus, $n \geq 12$. The smallest value for n is 12, making $3x + 4x + 6x = \dfrac{12 \cdot 13}{2} = 78$, so $x = 6$. Thus, $S(C) = 36$, $S(B) = 24$, and $S(A) = 18$. Since 1 + 2 + 3 + . . . + 8 = 36, it would appear that the smallest possible value for the largest element in C is 8, but the remaining elements {9, 10, 11, 12} cannot be partitioned into two sets with sums of 24 and 18. Try $C = \{2, 3, 4, 5, 6, 7, 9\}$, giving $B = \{1, 11, 12\}$ and $A = \{8, 10\}$, or $C = \{1, 5, 6, 7, 8, 9\}$, $B = \{3, 10, 11\}$ and $A = \{2, 4, 12\}$. These are partitions with the requisite sums. Thus, the answer is $\boxed{9}$.



T–6. Shown is half the octahedron inscribed in a sphere with center O. Without loss of generality, let the radius of the sphere equal 1. $AO = OB = 1$ so $BCDE$ is a square with side $\sqrt{2}$. The volume of the octahedron is twice the volume of pyramid A-$BCDE$ $= (2)\left(\frac{1}{3}\right)(2)(1) = \frac{4}{3}$. The volume of the sphere equals $\left(\frac{4}{3}\right)\pi(1)^3 = \frac{4\pi}{3}$. The ratio is $\frac{4\pi}{3} : \frac{4}{3} = \boxed{\pi \text{ or } \pi : 1}$.

T–7. Consider the 5 by 5 example in the problem. In order, as we spiral inward, the numbers are 1, 9, 17, 21, and 25. To get from 1 to 9 we walk 4 units to the right and 4 units down. To get from 9 to 17 we walk 4 units left and 4 units up and to the right. To get from 17 to 21 we walk 2 units to the right and 2 units down. To get from 21 to 25 we walk 2 units left and 2 units up and to the right. In other words, once we reach the upper left corner of a square of side n by n, the next number is reached in $2(n-2)$ steps and is, therefore, greater by $2(n-2)$, and the following number is again reached in $2(n-2)$ steps and is also greater by $2(n-2)$. We obtain the following column whose sum gives the desired result:

1	1
9	$1 + 2 \cdot 4$
17	$1 + 2 \cdot 4 + 2 \cdot 4$
21	$1 + 2 \cdot 4 + 2 \cdot 4 + 2 \cdot 2$
25	$1 + 2 \cdot 4 + 2 \cdot 4 + 2 \cdot 2 + 2 \cdot 2$

Adding vertically we obtain $5 \cdot 1 + 4(2 \cdot 4) + 3(2 \cdot 4) + 2(2 \cdot 2) + 2 \cdot 2 = 73$.

Applying this analysis to our problem we note that there are 27 numbers to be summed. They can be arranged in the following column:

$$\begin{array}{l}
1 \\
1 + 2(26) \\
1 + 2(26) + 2(26) \\
1 + 2(26) + 2(26) + 2(24) \\
1 + 2(26) + 2(26) + 2(24) + 2(24) \\
1 + 2(26) + 2(26) + 2(24) + 2(24) + 2(22) \text{ etc.}
\end{array}$$

Note that the number of 2(26)'s will be $26 + 25 = (2)(26) - 1 = 51$, the number of 2(24)'s will be $24 + 23 = 2(24) - 1 = 47$, etc. The sum of the 2(26)'s will be $2(26)(2 \cdot 26 - 1)) = (2 \cdot 26)^2 - 2 \cdot 26$, and a similar pattern will hold for all sums. Thus, the sum equals:

$27 + (2\cdot 26 - 1)(2\cdot 26) + (2\cdot 24 - 1)(2\cdot 24) + (2\cdot 22 - 1)(2\cdot 22) + \ldots + (2\cdot 2 - 1)(2\cdot 2)$ which is

$$27 + (2\cdot 26)^2 - 2\cdot 26 + (2\cdot 24)^2 - 2\cdot 24 + (2\cdot 22)^2 - 2\cdot 22 + \ldots + (2\cdot 2)^2 - 2\cdot 2 =$$

$$27 + 16\left(13^2 + 12^2 + \ldots + 1^2\right) - 4\left(13 + 12 + \ldots + 1\right) = 27 + 16\left(\frac{13\cdot 14\cdot 27}{6}\right) - 4\left(\frac{13\cdot 14}{2}\right) = \boxed{12767}.$$

Alternate solution: for n odd, the diagonal elements are given by the first line, their sum by the second:

$$1, 4n-3, 8n-15, 12n-35, \ldots, 2(n-1)n - \left((n-1)^2 - 1\right) = n^2, \ldots, 14n-49, 10n-25, 6n-9, 2n-1.$$

$$1 + (2n-1) + (4n-3) + (6n-9) + (8n-15) + (10n-25) + (12n-35) + (14n-49) + \ldots + 2(n-1)n - \left((n-1)^2 - 1\right)$$

Note that constants below the middle of the square are perfect squares, but starting in the upper left at $4n - 3$ and going to the middle to $2(n-1)n - \left((n-1)^2 - 1\right) = n^2$, the terms are 1 less than a perfect square. Thus, the number of terms 1 less than a perfect square is $\frac{n-1}{2}$. In creating a formula for the sum, we will subtract the perfect squares and then add back 1 times $\frac{n-1}{2}$. Thus, the sum equals:

$$1 + 2n(1 + 2 + 3 + \ldots + n - 1) - (1^2 + \ldots + (n-1)^2) + \frac{n-1}{2} = \frac{4n^3 - 3n^2 + 2n + 3}{6}.$$

Alternate solution: Let $T(k)$ be the sum of the diagonal elements in the spiral whose square's side is $2k - 1$. Here $T(1) = 1, T(2) = 1 + 5 + 9 = 15$ and $T(3) = 1 + 9 + 17 + 21 + 25 = 73$. The problem asks for $T(14)$. The square of side $2k + 1$ can be obtained from the square of side $2k - 1$ by adding $8k$ to each element and then wrapping a single layer containing $1, 2, \ldots, 8k$ around the square of side $2k - 1$, putting 1 and $4k + 1$ in the corners of the diagonal. Thus, from $T(2) = 1 + 9 + 5$, we obtain

$$T(3) = 1 + (1 + 16) + (9 + 16) + (5 + 16) + (4\cdot 2 + 1) = T(2) + 3\cdot 16 + 1 + (4\cdot 2 + 1).$$

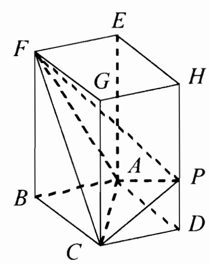
In general, $T(k+1) = T(k) + (2k - 1)8k + 1 + (4k + 1)$. Thus,

$T(k+1) - T(k) = 16k^2 - 4k + 2$. So, $T(14) = \sum_{k=0}^{13} T(k+1) - T(k) = 1 + \sum_{k=1}^{13} 16k^2 - 4k + 2 =$

$$1 + 16\left(\frac{13\cdot 14\cdot 27}{6}\right) - 4\left(\frac{13\cdot 14}{2}\right) + 2(13) = 12767.$$

T–8. Consider the set $S_{2000} = \{1, 2, 3, \ldots, 2000\}$ whose elements are the powers of 2. If the common ratio is 2, then the desired sets of exponents are 1-2-3, 2-3-4, . . . , 1998-1999-2000. First elements run from 1 to 1998 so there are 1998 sets of three elements forming an increasing geometric progression with a common ratio of 2. The largest first element gives the number of sets and equals 2000 – 3 + 1 = 1998. If the common ratio is 2^2, then the desired sets of exponents are 1-3-5, 2-4-6, . . . , 1996-1998-2000. The largest first element is 2000 – 5 + 1 = 1996. If the common ratio is 2^3, the sets are 1-4-7, . . . , 1994-1997-2000 and the largest first element is 2000 – 7 + 1 = 1994. The largest ratio is 2^{999} and its triples are 1-1000-1999 and 2-1001-2000. The largest first element is 2000 – 1999 + 1 = 2. Thus, the number of sets of three is 1998 + 1996 + 1994 + . . . + 6 + 4 + 2 = $\boxed{999{,}000}$.

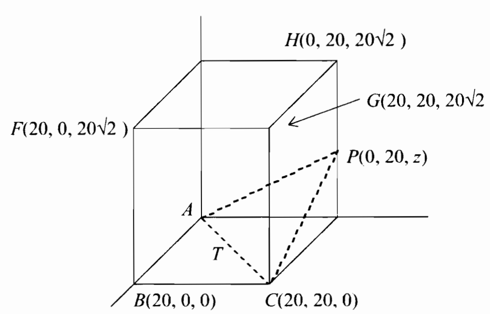
T–9. Since ΔGAC is a 45–45–90 right triangle and $AC = 20\sqrt{2}$, then GC = the height of the box $= 20\sqrt{2}$. Since both $\overline{BH}$ and the altitude from P to $\overline{AC}$ rise at a 45° angle, PD = half the height or $10\sqrt{2}$. The volume of the box is $20(20)20\sqrt{2} = 8000\sqrt{2}$. The volume of pyramid $PACD$ is $\left(\frac{1}{3}\right)\left(\frac{1}{2}\cdot 20\cdot 20\right)10\sqrt{2}$



$= \dfrac{2000\sqrt{2}}{3}$. The volume of $FBAC = \dfrac{4000\sqrt{2}}{3}$ since its height is twice that of $PACD$. Using $CPHG$ as the base, the height of $FCPHG$ is $FG = 20$ and its volume is $\left(\frac{1}{3}\right)20\left(20\cdot 20\sqrt{2}-\left(\frac{1}{2}\cdot 20\cdot 10\sqrt{2}\right)\right) = 2000\sqrt{2}$.

Using $PHEA$ as the base and $FE = 20$ as the height, the volume of $FPHEA$ similarly equals $2000\sqrt{2}$. Thus, the volume of $FPCA = 8000\sqrt{2} - \dfrac{2000\sqrt{2}}{3} - \dfrac{4000\sqrt{2}}{3} - 2\left(2000\sqrt{2}\right) = \boxed{2000\sqrt{2}}$.

<u>Alternate solution</u>: in the above diagram, let T be the midpoint of $\overline{AC}$, giving $T(10, 10, 0)$. Since vectors $\overrightarrow{BH} = \langle -20, 20, 20\sqrt{2}\rangle$ and $\overrightarrow{TP} = \langle -10, 10, z\rangle$ are parallel, $z = 10\sqrt{2}$. Using the cross product or the dot product, we obtain $\langle \sqrt{2}, -\sqrt{2}, 2\rangle$ for the normal vector to plane PAC.



The equation of plane PAC is therefore $x\sqrt{2} - y\sqrt{2} + 2z = 0$. The distance from F to the plane equals $\dfrac{\left|20\sqrt{2} - 0\sqrt{2} + 2 \cdot 20\sqrt{2}\right|}{\sqrt{2+2+4}} = 30$. Since $PA = PC = 10\sqrt{6}$ and $AC = 20\sqrt{2}$, the height of $\Delta PAC = 20$ and its area is $200\sqrt{2}$, giving a volume of $2000\sqrt{2}$.

T–10. Think of the top node as the root and the other nodes as its descendants, each of which has exactly one parent and 0 or 2 children. Let b_k be the number of subtrees rooted at a kth generation node. Then $b_4 = 1$ for the 8 fourth generation nodes. The third generation nodes have two children each with $b_4 = 1$ subtrees rooted there. Accounting for the empty subtree, we have $b_3 = (1+1)^2 = 4$ possibilities. Similarly, the second generation nodes have $b_2 = (4+1)^2 = 25$ subtrees and the root has $b_1 = (25+1)^2 = 676$ subtrees. The total number of subtrees equals $1 \cdot 676 + 2 \cdot 25 + 4 \cdot 4 + 8 \cdot 1 = \boxed{750}$.

Alternate solution: Let $f(n)$ be the number of connected subtrees of a complete binary tree on 2^{n-1} nodes and let $g(n)$ be the number of connected subtrees of a complete binary tree on 2^{n-1} nodes which include the root (top) node. Then $f(1) = g(1) = 1$ since the only connected subtree of a single node is that node itself. We will find recursive relations for f and g. First, note that $g(n+1) = \left(g(n)+1\right)^2$ because any subtree containing the root node must have a connected subtree on either side of the root which is either empty (there is 1 of these) or contains the top node (there are $g(n)$ of these). Next note that $f(n+1) = g(n+1) + 2f(n)$ since any subtree included in the count of $f(n+1)$ must either have a tope node or be one of the trees counted by g of which there are $g(n+1)$. Or the binary tree does not have a top node and therefore must consist of one of the right or left subtrees empty (otherwise the tree would not be connected) and the other subtree (left or right respectively) containing all the vertices of the graph. Since the graph can therefore go in either the left or right (smaller) subtree, there are $2f(n)$ such graphs. So, $g(2) = 4$, $g(3) = 25$, $g(4) = 676$ from the first relation and $f(2) = g(2) + 2f(1) = 4 + 2 = 6$, $f(3) = g(3) + 2f(2) = 25 + 2(6) = 37$, and $f(4) = g(4) + 2f(3) = 676 + 2(37) = 750$.

Alternate Solution: number the vertices along the left side top to bottom 3, 2, 1, and let $h(n)$ be the number of subtrees whose highest vertex is n. Note that there is another node whose subtree structures below it will be the same as vertex 2, and that three other subtree structures will be the same as that of vertex 1. When $n > 0$, the subtree may consist of: 1) only vertex n, or 2) of the left (or right) edge leading down from it together with a subtree structure starting at vertex $n-1$, or 3) the vertex equivalent to it, or 4) both the left and right edge and any combination of substructures starting at $n-1$ or its equivalent. Thus,

$h(n) = 1 + 2h(n-1) + \left(h(n-1)\right)^2$ when $n > 0$. Note that $h(0) = 1$ and the formula gives

$h(1) = 1 + 2h(0) + \left(h(0)\right)^2 = 1 + 2 + 1 = 4$, $h(2) = 1 + h(1) + \left(h(1)\right)^2 = 1 + 8 + 16 = 25$, and

$h(3) = 1 + 2h(2) + \left(h(2)\right)^2 = 1 + 50 + 625 = 676$. The total number of subtrees equals

$h(3) + 2h(2) + 4h(1) + 8h(0) = 750$.

Let $T = T(a, b, c)$ be ΔABC whose sides $BC = a$, $CA = b$, and $AB = c$ satisfy both $0 < a \le b \le c$ and $a + b > c$.

Define $T^2 = T(a^2, b^2, c^2)$ to be the square of triangle T, provided that such a triangle exists.

Define $\sqrt{T} = T\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$ to be the square root of triangle T, provided that such a triangle exists.

Use $K(T)$ for the area of triangle T and $P(T)$ for the perimeter of triangle T.

Results from a preceding problem may be used in a succeeding problem but not vice versa.

You may use Fermat's Last Theorem as a reason in any proof.

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1. a) Show that the square of an equilateral triangle is equilateral.

   b) Prove that the square of a right triangle does not exist.

2. Compute all x for which the area of $T = T(1, 1, x)$ is the same as the area of T^2.

3. a) Prove that T^2 exists if and only if T is acute.

   b) Prove that $\sqrt{T}$ exists for all T.

4. a) Prove that all the angles of $\sqrt{T}$ are acute.

   b) Prove that $\cos C \le \frac{1}{2}$ with equality if and only if T is equilateral.

5. Prove that the largest angle of T is at least as close to 60° as the largest angle of T^2.

Let $X_n = T(n - 1, n, n + 1)$ for n a real number.

6. a) Determine with proof, all n for which $\left(X_n\right)^2$, i.e., the square of X_n, is a triangle.

   b) Note that $X_{11} = T(10, 11, 12)$ can be squared, producing triangle $T(100, 121, 144)$ and that this can be squared again, producing triangle $T(10000, 14641, 20736)$. The latter, however, cannot be squared. Now for an integer p, there exists a real number k, $p \le k \le p + 1$, such that for all $n \ge k$, X_n can be squared at least twice. Find p and justify your answer.

7. a) Provide, with reasons, a triangle which can be squared at least 2001 times and at most 2001 times.

b) Let $N(T)$ be the number of times a triangle T can be squared. For example, $N\left(X_{1\,1}\right) = 2$. Determine, with proof, all triangles T for which $N(T) = \infty$. Specify side relationships and angle restrictions.

8. a) Prove that if $n = \sqrt{a} + \sqrt{b} + \sqrt{c}$ for integers n, a, b, and c, then a, b, and c must be perfect squares.

b) Using (a) or otherwise, prove that if $T = T(a, b, c)$ is a right triangle with <u>integer</u> sides, then the perimeter of $\sqrt{T}$ cannot be an integer.

9. Prove that there exists a right triangle T with the same perimeter as $\sqrt{T}$.

10. Define the reciprocal of $T(a, b, c)$ to be $T^{-1} = T\left(\frac{1}{c}, \frac{1}{b}, \frac{1}{a}\right)$ where $\frac{1}{c} < \frac{1}{b} < \frac{1}{a}$.

When T^{-1} is itself a triangle, define T to be <u>invertible</u>. Show that if T is invertible,

then $a > \left(\frac{3-\sqrt{5}}{2}\right)c$.

1. a) If $T = T(a,a,a)$, then $T^2 = T(a^2,a^2,a^2)$ and T^2 is clearly equilateral.

   b) If $T = T(a,b,c)$ is a right triangle, then $a^2 + b^2 = c^2$ and so $T^2 = T(a^2,b^2,c^2)$ fails the triangle inequality.

2. Using Heron's formula, $K(T) = \sqrt{\left(1+\frac{x}{2}\right)\left(1-\frac{x}{2}\right)\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)} = \frac{x}{4}\sqrt{4-x^2}$ and $K(T^2) = \frac{x^2}{4}\sqrt{4-x^4}$.

   Setting these equal we obtain $\sqrt{4-x^2} = x\sqrt{4-x^4} \rightarrow x^6 - 5x^2 + 4 = 0$. Let $u = x^2$ giving $u^3 - 5u + 4 = 0$. By inspection, by using the rational root theorem and synthetic division, or by realizing that the equilateral triangle $T(1, 1, 1)$ solves the problem, we know that $u = 1$ is one solution. This gives $(u-1)(u^2+u-4) = 0$, making $u = 1$ or $\frac{\sqrt{17}-1}{2}$. Thus, $x = 1$ or $\sqrt{\frac{\sqrt{17}-1}{2}} = \frac{1}{2}\sqrt{2\sqrt{17}-2}$.

3. a) Given T, then T^2 is a triangle iff the sides satisfy the triangle inequality, namely that $a^2 + b^2 > c^2$. By the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos C$ giving $c^2 + 2ab\cos C = a^2 + b^2$. Thus, $a^2 + b^2 > c^2$ iff $2ab\cos C > 0$ which is true whenever $0 < m\angle C < 90°$. Since neither $\angle A$ nor $\angle B$ is greater than $\angle C$, then T is acute-angled.

   b) Given T, then $\sqrt{T}$ is a triangle iff $\sqrt{a} + \sqrt{b} > \sqrt{c}$. Since $\left(\sqrt{a}+\sqrt{b}\right)^2 =$ $a + 2\sqrt{ab} + b > a + b > c$, taking the square root yields $\sqrt{a} + \sqrt{b} > \sqrt{c}$.

4. a) Let θ be the angle opposite the longest side $\sqrt{c}$. Then $\left(\sqrt{c}\right)^2 = \left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 - 2\sqrt{a}\sqrt{b}\cos\theta \rightarrow$ $\cos\theta = \frac{a+b-c}{2\sqrt{ab}}$. Since $a + b > c$, then $\cos\theta > 0$, making $\sqrt{T}$ an acute-angled triangle.

   b) Since $m\angle A$ and $m\angle B$ are both smaller than $m\angle C$, we have $180° = m\angle A + m\angle B + m\angle C \leq 3\,m\angle C$. Thus, $m\angle C \geq 60°$ with equality when $m\angle A = m\angle B = m\angle C$.

5. Given $m\angle C \geq m\angle B \geq m\angle A$, let $\cos C_X$ stand for the cosine of $\angle C$ in triangle X. Then we wish to show that $\cos C_{\sqrt{T}} \geq \cos C_T$. To avoid ugly expressions, let $T = T(a^2, b^2, c^2)$ so that $\sqrt{T} = T(a, b, c)$. By the Law of Cosines:

$$\cos C_{\sqrt{T}} - \cos C_T = \frac{a^2 + b^2 - c^2}{2ab} - \frac{a^4 + b^4 - c^4}{2a^2b^2}$$

$$= \frac{a^3b + ab^3 - abc^2 - a^4 - b^4 + c^4}{2a^2b^2}$$

$$= \frac{\left(c^2 + b^2 - ab\right)\left(c^2 - b^2\right) + a^3\left(b - a\right)}{2a^2b^2} \geq 0 \text{ since } c \geq b \geq a.$$

Note: From (4b) we know that the largest angle $C \geq 60°$. Thus, #5 proves that the largest angle of $\sqrt{T}$ is basically closer to 60° than the largest angle of T. If one also proved that the smallest angle of $\sqrt{T}$ is at least as close to 60° as the smallest angle of T, then one would have shown that $\sqrt{T}$ is closer to being equilateral than T.

6. a) The square of X_n is $\left((n-1)^2, n^2, (n+1)^2\right)$ and for a triangle to exist, $(n-1)^2 + n^2 > (n+1)^2 \rightarrow$ $n^2 - 4n > 0 \rightarrow n > 4$. But we should check to see if X_n is a triangle for all $n > 4$. Here we find that $n - 1 + n > n + 1 \rightarrow n > 2$, so $n > 4$.

b) The square of $\left(X_n\right)^2$ is $\left((n-1)^4, n^4, (n+1)^4\right)$. We have $(n-1)^4 + n^4 > (n+1)^4 \rightarrow$ $n^4 - 8n^3 - 8n > 0 \rightarrow n^3 - 8n^2 - 8 > 0$. Let $R(n) = n^3 - 8n^2 - 8$. Since $R(8) = -8$ and $R(9) = 73$ and R is continuous, there is a zero between 8 and 9. For completeness we must verify that $R(n)$ isn't negative for some larger value of n. For $n > 9$, $R(n) = n^3 - 8n^2 - 8 > n^3 - 8n^2 - n^2 = n^2(n-9)$ and the last is greater than 0 for all $n > 9$. Thus, $p = 8$.

7. a) $T = T(3, 4, 5)$ is a triangle and since it is a right triangle, we know by (1b) that T^2 doesn't exist. From (3b) we know that if a triangle exists, then its square root exists. So, starting with $T = T(3, 4, 5)$, take successive square roots 2001 times. This gives $T = T\left(3^{1/2^{2001}}, 4^{1/2^{2001}}, 5^{1/2^{2001}}\right)$, a triangle which can be squared exactly 2001 times.

b) It seems reasonable that if $N(T) = \infty$, then $c = b$, for if $c > b \geq a$, the continual squaring of c would eventually make c much larger than the sum of a and b, and the squaring process would eventually end. So let $T = T(a, b, b)$ and check the triangle inequality. Clearly, $a^{2^k} + b^{2^k} > b^{2^k}$ and $b^{2^k} + b^{2^k} > a^{2^k}$ for all k iff $b \geq a$. Thus, $N(T) = \infty$ iff T is an isosceles triangle with base angles $\geq 60°$.

8. a) Suppose $\sqrt{p} + \sqrt{q} = m$ for integers p, q, and m. Then $p = \left(m - \sqrt{q}\right)^2 = m^2 - 2m\sqrt{q} + q$ and q has to be a perfect square in order for both sides to be integers. Similarly, p must be a perfect square. Call this result #1. Now, squaring $\sqrt{a} + \sqrt{b} = n - \sqrt{c}$ gives $2\sqrt{ab} + 2n\sqrt{c} = n^2 + c - a - b$. The right side is an integer and the left side equals $\sqrt{4ab} + \sqrt{4n^2c}$ and by #1, both $\sqrt{4ab}$ and $\sqrt{4n^2c}$ must be integers. Thus, $2n\sqrt{c}$ is an integer, making c a perfect square. Returning to $\sqrt{a} + \sqrt{b} = n - \sqrt{c}$, the right side is an integer, so by #1, both a and b must be perfect squares.

b) Suppose the perimeter of $\sqrt{T} = T\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$ is an integer. Then by (8a) $a = x^2$, $b = y^2$, and $c = z^2$ for integers x, y, and z. Since a, b, and c satisfy $a^2 + b^2 = c^2$, then $(x^2)^2 + (y^2)^2 = (z^2)^2$ giving $x^4 + y^4 = z^4$, a false result. So the perimeter of $\sqrt{T}$ can't be integral.

9. If the sides of T are all greater than 1, then each side is greater than its square root so $P(T) > P\left(\sqrt{T}\right)$. If the sides of T are all less than 1, then each side is less than its square root so $P(T) < P\left(\sqrt{T}\right)$. To obtain equality of perimeters, the smallest side should be less than 1 and the largest greater than 1. We ought to obtain the simplest solution if the middle side equals 1. So, let $a = x < 1$, $b = 1$, and $c = \sqrt{1 + x^2}$.

Then $T = T\left(x, 1, \sqrt{1 + x^2}\right)$ and $\sqrt{T} = T\left(\sqrt{x}, 1, \sqrt[4]{1 + x^2}\right)$. Let $f(x) = P(T) - P(\sqrt{T}) =$ $\left(x - \sqrt{x}\right) + \left(\sqrt{1 + x^2} - \sqrt[4]{1 + x^2}\right)$. This is clearly a continuous function. Since $f(.7) = -.0208$ and $f(.8) = .05455$, the Intermediate Value Theorem assures us that at some value of x between .7 and .8, the two perimeters are equal and we have shown the existence of a right triangle whose perimeter equals that of its square root. That value of x is approximately .72864. Note: b doesn't need to equal 1, but it seems reasonable that it can't grow too large. What is the maximum value that b can take on or what is its least upper bound?

Alternate solution: Consider right triangle $T_1 = \left(1, 1, \sqrt{2}\right)$ with $\sqrt{T_1} = \left(1, 1, \sqrt[4]{2}\right)$. Then $P(T_1) = 2 + \sqrt{2}$ and $P\left(\sqrt{T_1}\right) = 2 + \sqrt[4]{2}$ and clearly $P\left(T_1\right) - P\left(\sqrt{T_1}\right) > 0$. Divide the terms of T_1 by 4, producing right triangle $T_4 = \left(\frac{1}{4}, \frac{1}{4}, \frac{\sqrt{2}}{4}\right)$ with $\sqrt{T_4} = \left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt[4]{2}}{2}\right)$. We have $P\left(T_4\right) = \frac{2 + \sqrt{2}}{4}$ and $P\left(\sqrt{T_4}\right) = \frac{2 + \sqrt[4]{2}}{2}$.

Note that $P\left(T_4\right) < P\left(\sqrt{T_4}\right)$, since otherwise $\frac{2 + \sqrt{2}}{4} \geq \frac{2 + \sqrt[4]{2}}{2} \rightarrow 2 + \sqrt{2} \geq 4 + 2\sqrt[4]{2} \rightarrow$ $\sqrt{2} - 2 \geq \sqrt[4]{2}$, clearly a contradiction. Thus, $P\left(T_4\right) - P\left(\sqrt{T_4}\right) < 0$. Now consider the function $f(x) = P\left(T_x\right) - P\left(\sqrt{T_x}\right)$ where x is the number dividing the original terms of T_1. $T_x = T\left(\frac{1}{x}, \frac{1}{x}, \frac{\sqrt{2}}{x}\right)$ and $\sqrt{T_x} = \left(\frac{1}{\sqrt{x}}, \frac{1}{\sqrt{x}}, \frac{\sqrt[4]{2}}{\sqrt{x}}\right)$. Thus, $f(x) = \frac{2 + \sqrt{2}}{x} - \frac{2 + \sqrt[4]{2}}{\sqrt{x}} = \frac{(2 + \sqrt{2}) - (2 + \sqrt[4]{2})\sqrt{x}}{x}$. We have just seen that $f(1) > 0$ and $f(4) < 0$. Since f is continuous for $x > 0$, then by the Intermediate Value Theorem, there must be a value of x such that $f(x) = 0$. Here, $x \approx 1.1460826$ and the triangle is approximately $T = T(.872537, .872537, 1.2339)$.

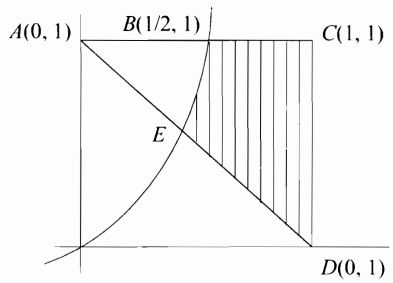
10. For $T(a, b, c)$ we have $0 < a \leq b \leq c$ and $a + b > c$. If T^{-1} is a triangle, then we have $0 < \frac{1}{c} \leq \frac{1}{b} \leq \frac{1}{a}$ and $\frac{1}{b} + \frac{1}{c} > \frac{1}{a}$. From $a + b > c$, we obtain $\frac{a}{c} + \frac{b}{c} > 1$ (#1). From $\frac{1}{b} + \frac{1}{c} > \frac{1}{a}$, we obtain $\frac{c}{b} + 1 > \frac{c}{a}$ (#2). Let $x = \frac{a}{c}$ and $y = \frac{b}{c}$, resulting in $0 < x \leq y \leq 1$ (#3). Then (#1) gives $x + y > 1$ and (#2) gives $\frac{1}{y} + 1 > \frac{1}{x} \rightarrow y < \frac{x}{1 - x}$.

Shown at the right are the graphs of these inequalities:

$0 < x \le y \le 1$

$x + y > 1$

$y < \dfrac{x}{1-x}$



The shaded region $EBCD$ represents the solution set.

Only the horizontal and vertical boundaries are included:

To find the coordinates of point E solve the system $x + y = 1$ and $y = \dfrac{x}{1-x}$, obtaining $x^2 - 3x + 1 = 0$

$\rightarrow x = \dfrac{3 \pm \sqrt{5}}{2}$. Reject $\dfrac{3+\sqrt{5}}{2}$ since it exceeds 1. Then for $x > \dfrac{3-\sqrt{5}}{2}$, T is invertible. Since $x = \dfrac{a}{c}$,

we have $\dfrac{a}{c} > \dfrac{3-\sqrt{5}}{2} \rightarrow a > \left(\dfrac{3-\sqrt{5}}{2}\right)c$.

Note that $a > \dfrac{c}{\tau^2}$ if we let τ stand for the golden ratio, $\dfrac{1+\sqrt{5}}{2}$.

<u>This Power Question is full of great research possibilities. Here are two additional problems:</u>

1. Let T be a right triangle. With proof, compute the largest value of cos C for $\sqrt{T}$.

2. Let T be a <u>scalene</u> triangle in which exactly one side has a length of 1. If T and T^2 have exactly two sides of the same length, then compute the least upper bound for the area of T. Show your work and justify your answer.

I–1. If $\frac{4}{2001} < \frac{a}{a+b} < \frac{5}{2001}$, compute the number of integral values that $\frac{b}{a}$ can take on.

I–2. A lattice rectangle has vertices at lattice points and sides parallel to the axes. It contains all lattice points in its interior and on its sides. If the shorter side of lattice rectangle R is doubled, the new rectangle contains 304 more lattice points than R. Compute the minimum possible area of R.

I–3. Compute the largest real value for b such that the solutions to the following equation are integers:

$$\left(\log_{2^{10}} x^{2b}\right)^2 = \log_{2^{10}} x^4.$$

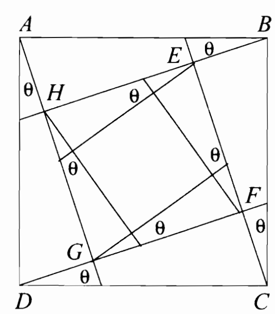
I–4. The vertices of a regular dodecagon are given by (x_i, y_i) for $i = 1, 2, \ldots, 12$ in clockwise order.

If $(x_1, y_1) = (15, 9)$ and $(x_7, y_7) = (15, 5)$, compute $\sum_{i=1}^{12} (x_i - y_i)$.

I–5. For $m > 0$, $f(x) = mx$. The acute angle θ formed by the graphs of f and f^{-1} is such that $\tan\theta = \frac{5}{12}$.

Compute the sum of the slopes of f and f^{-1}.

I–6. In unit square $ABCD = S_0$, two pairs of parallel lines are drawn, each making an angle of θ with a side of S_0. These lines produce an inner square, $EFGH = S_1$. The procedure is repeated using S_1 and angle θ, producing an inner square S_2. The process is continued. If $\theta = 15°$, compute the sum of the areas of squares $S_0, S_1, S_2, S_3, \ldots$



I–7. Let $[x]$ represent the greatest integer less than or equal to x. Compute the number of first quadrant ordered pair (x, y) solutions to the following system:

$$\begin{aligned} x + [y] &= 5.3 \\ y + [x] &= 5.7 \end{aligned}$$

I–8. For $0 < x < 1$, let $f(x) = (1+x)(1+x^4)(1+x^{16})(1+x^{64})(1+x^{256})\ldots$

Compute: $f^{-1}\left(\frac{8}{5f\left(\frac{3}{8}\right)}\right)$.

1. 100

2. 288

3. $\sqrt{10}$

4. 96

5. $\frac{13}{6}$ or $2.1\overline{6}$

6. 2

7. 6

8. $\frac{9}{64}$

I–1. $\frac{2001}{4} > \frac{a+b}{a} > \frac{2001}{5} \to 500.25 > 1 + \frac{b}{a} > 400.2 \to 499.25 > \frac{b}{a} > 399.2.$

Thus, $400 \le \frac{b}{a} \le 499$, giving $\boxed{100}$ integral values for $\frac{b}{a}$.

I–2. If the sides are n and m with $n > m$, then the number of points on each side is $n + 1$ and $m + 1$. If the side of length m is doubled, the number of points on each side is $n + 1$ and $2m + 1$. Thus,

$(2m+1)(n+1) - (m+1)(n+1) = 304 \to mn + m = m(n+1) = 2^4 \cdot 19$. If $m = 1$ and $n + 1 = 304$, there is no rectangle, so consider these 4 pairs of $(m, n+1)$: (2, 152), (4, 76), (8, 38), (16, 19). The areas are respectively $2 \cdot 151 = 302$, $4 \cdot 75 = 300$, $8 \cdot 37 = 296$, and $16 \cdot 18 = 288$. Answer: $\boxed{288}$.

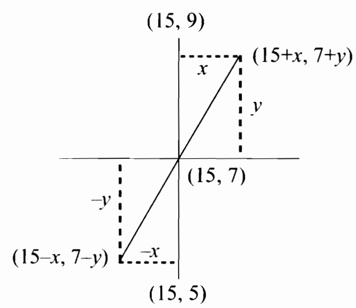
I–3. $\left(2b\log_{2^{10}} x\right)^2 = 4\log_{2^{10}} x \to 4b^2\left(\log_{2^{10}} x\right)^2 - 4\log_{2^{10}} x = 0 \to \left(\log_{2^{10}} x\right)\left(b^2\log_{2^{10}}(x) - 1\right) = 0$. If

$\log_{2^{10}} x = 0$, then $x = 1$. If $b^2\log_{2^{10}}(x) - 1 = 0$, then $\log_{2^{10}} x = \frac{1}{b^2}$, giving $x = \left(2^{10}\right)^{1/b^2} = 2^{10/b^2}$.

There are several values for b that make x an integer. If $b = \pm 1$, then $x = 2^{10}$. If $b = \pm\frac{\sqrt{10}}{3}$, then

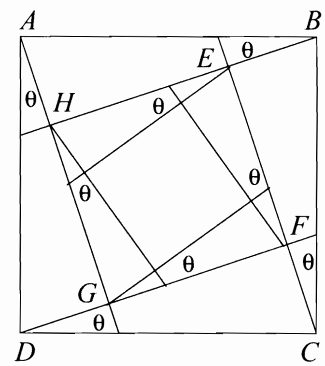
$x = 2^9$. But the largest value of b making the second solution an integer is $b = \boxed{\sqrt{10}}$ which gives $x = 2^1$.

I–4. Given opposite vertices (15, 9) and (15, 5), the center of the dodecagon is (15, 7). By congruent triangles, note that for any vertex $(15 + x, 7 + y)$ there is a corresponding opposite vertex $(15 - x, 7 - y)$. The sum of all 12 x-coordinates equals $15 \cdot 12$ and the sum of all 12 y-coordinates $= 7 \cdot 12$. The difference is $(15 - 7)12 = \boxed{96}$.



I–5. If $f(x) = mx$, then $f^{-1}(x) = \frac{x}{m}$. Using $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$ gives $\tan\theta = \frac{m - \frac{1}{m}}{1 + m\left(\frac{1}{m}\right)} = \frac{m^2 - 1}{2m} = \frac{5}{12}$.

Thus, $6m^2 - 5m - 6 = 0 \to (3m + 2)(2m - 3) = 0 \to m = \frac{3}{2}, \frac{1}{m} = \frac{2}{3}$, making the sum equal $\boxed{\frac{13}{6}}$.



I–6. The first 2 pairs of parallel lines divide the unit square S_0 into S_1 and the four congruent triangles *DFC*, *CEB*, *BHA*, and *AGD*. Consider ΔDFC. Since $DC = 1$, $DF = \cos\theta$ and $FC = \sin\theta$, its area is $\frac{1}{2}(\cos\theta)(\sin\theta)$, and the sum of the areas of all 4 triangles is $2\sin\theta\cos\theta = \sin 2\theta$. Since $\theta = 15°$, the sum of the areas is $\sin 30° = \frac{1}{2}$. Thus, the area of $S_1 = 1 - \frac{1}{2} = \frac{1}{2}$. Similarly, the area of $S_2 = \frac{1}{4}$, the area of $S_3 = \frac{1}{8}$ and the desired sum is $1 + \frac{1}{2} + \frac{1}{4} + \ldots = \boxed{2}$.

I–7. Let $x = n + 0.3$ and $y = m + 0.7$ for non-negative integers *m* and *n*. Then $m + n = 5$ and there are 6 pairs of integers from (0, 5) to (5, 0) for which this is true. The solutions are (0.3, 5.7), (1.3, 4.7), (2.3, 3.7), (3.3, 2.7), (4.3, 1.7), and (5.3, 0.7). Ans: $\boxed{6}$.

I–8. If $f(x) = (1+x)(1+x^4)(1+x^{16})(1+x^{64})\cdots$, then $f(x^2) = (1+x^2)(1+x^8)(1+x^{32})(1+x^{128})\cdots$ and $f(x)f(x^2) = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\cdots = 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x}$. Thus, $f(x^2) = \frac{1}{f(x)(1-x)}$. Letting $x = \frac{3}{8}$, we obtain $f\left(\frac{9}{64}\right) = \frac{1}{f\left(\frac{3}{8}\right)\left(1-\frac{3}{8}\right)} = \frac{8}{5f\left(\frac{3}{8}\right)}$. Thus, $\frac{9}{64} = f^{-1}\left(\frac{8}{5f(3/8)}\right)$. Answer: $\boxed{\frac{9}{64}}$.

<u>More rigorously</u>, let $g(x) = f(x)f(x^2) = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots$. We claim that $g(x) = \frac{1}{1-x}$ because $g(x)(1-x) = (1-x)\left((1+x)(1+x^2)(1+x^4)(1+x^8)\cdots\right) = (1-x^2)(1+x^2)(1+x^4)(1+x^8)\cdots = (1-x^4)(1+x^4)(1+x^8)\cdots = (1-x^n)g(x^n)$. For $0 < x < 1$, each expression approaches 1 as *n* gets infinitely large, so $g(x)(1-x) = 1$. Therefore, $f\left(\frac{3}{8}\right)f\left(\frac{9}{64}\right) = \frac{1}{1-(3/8)} = \frac{8}{5}$, so $f\left(\frac{9}{64}\right) = \frac{8}{5f(3/8)}$.

Therefore, $f^{-1}\left(\frac{8}{5f(3/8)}\right) = \frac{9}{64}$.

ARML Relay #1 – 2001

R1–1. The area of a regular octagon is $2 + 2\sqrt{2}$. Compute the length of a side.

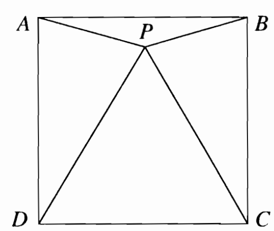
R1–2. Let T = TNYWR. Compute the number of real roots to $x^3 + (T + 1)x^2 + (T + 1)x + 1 = 0$.

R1–3. Let T = TNYWR. Compute: $(\log_3 2)(\log_8 27) + (\log_3 2)(\log_4 5)(\log_{25} T)$.

ARML Relay #2 – 2001

R2–1. Compute the smallest positive value of x in degrees such that $\sin x = \cos\left(x^2\right)$.

R2–2. Let T = TNYWR. $ABCD$ is a square with $PD = PC = DC$. The area of $\Delta PBC = T$. Compute the sum of the areas of ΔPAB and ΔPDC.



R2–3. Let T = TNYWR. Compute the number of lattice points which lie strictly inside the triangle formed by $x = 0, y = 0$, and $x + y = T$.

Relay #1:

R1–1. 1

R1–2. 1

R1–3. 1

Relay #2:

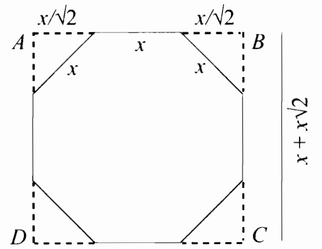
R2–1. 9

R2–2. 18

R2–3. 136

Solutions to the ARML Relay #1 – 2001

R1–1. Let the side of the octagon be x. The area of the octagon equals the area of square $ABCD$ minus the 4 corner triangles:



$$\left(x + x\sqrt{2}\right)^2 - 4 \cdot \frac{1}{2}\left(\frac{x}{\sqrt{2}}\right)^2 = 2 + 2\sqrt{2}.\text{ Thus,}$$

$$2x^2 + 2x^2\sqrt{2} = 2 + 2\sqrt{2} \rightarrow x = \boxed{1}.$$

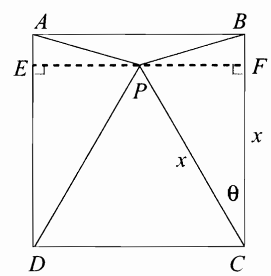
R1–2. By inspection, $x = -1$ is a solution, giving $(x + 1)\left(x^2 + Tx + 1\right) = 0$. The discriminant is $T^2 - 4$. If $T < -2$ or $T > 2$, the quadratic has 2 distinct solutions giving 3 real roots. If $T = 2$, the quadratic has a double solution of $-1 \rightarrow$ the cubic's three solutions are -1, giving 1 real root. If $T = -2$, the solution to the quadratic is 1, giving 2 real roots, 1 and -1. If $-2 < T < 2$, the quadratic has no real roots, making -1 the only real solution of the cubic. Since $T = 1$, the cubic has just $\boxed{1}$ real root.

R1–3. We have $\frac{\log 2}{\log 3} \cdot \frac{\log 3^3}{\log 2^3} + \frac{\log 2}{\log 3} \cdot \frac{\log 5}{2\log 2} \cdot \frac{\log T}{2\log 5} = 1 + \frac{\log T}{4\log 3}$. Since $T = 1$, $\log T = 0$ and the answer is $\boxed{1}$.

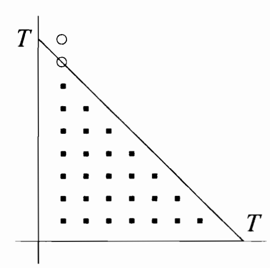
Solutions to the ARML Relay #2 – 2001

R2–1. $\sin x = \sin\left(90° - x^2\right) \rightarrow x = 90° - x^2 \rightarrow x^2 + x - 90° = 0 \rightarrow (x + 10°)(x - 9°) = 0 \rightarrow x = \boxed{9°}$.

R2–2. Since $PE = PF = \frac{1}{2}AB$ and $AD = BC$, the areas of ΔPBC and ΔPAD are both equal to $T = 9$. Since the sum of their areas equals $2\left(\frac{1}{2}\right)\left(\frac{AB}{2}\right)BC = \frac{1}{2}(AB)(BC)$ equals half the area of the square equals $2T = 18$, the sum of the areas of the other two triangles is half the area of the square which is $\boxed{18}$. This solution does not require that ΔPDC be equilateral, only isosceles.



R2–3. By inspection the sum is $(T-2) + (T-3) + (T-4) + \ldots + 1 = \frac{(T-2)(T-1)}{2}$. Since $T = 18$, the answer is $\frac{16 \cdot 17}{2} = \boxed{136}$.



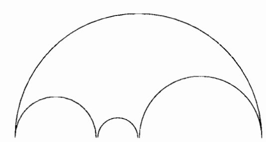
Note: Pass answers from position 1 to 8 and from position 15 to 8.

1. A square has side of length s and a regular hexagon has side of length h. The hexagon's area is three times that of the square. Compute $\frac{s^4}{h^4}$.

2. Let T = TNYWR. Compute $\log_8\left(2^{24}\right)^T$.

3. Let T = TNYWR. If $(x^2 - Nx + 7)(2x^2 + Tx + 9)(5x^3 + Tx + 13) = 10x^7 + P(x)$ where P is a fifth degree polynomial in x, compute $\frac{N}{T}$.

4. Let T = TNYWR. Shown are four semicircular arcs whose centers are collinear. The radius of the largest semicircular arc is T. Let P be the perimeter of the entire figure. Compute $\frac{P}{\pi}$.



5. Let T = TNYWR. In ΔABC, $AB = AC = 5$, and $BC = 2T$. Compute $\sin\angle BAC$.

6. Let T = TNYWR and let $K = 25T$. A line passes through $A(0, K)$ with slope $-K$. Compute the area of the triangle formed by the origin and the x- and y-intercepts of the line.

7. Let T = TNYWR and let $K = T^2 - 4$. In the arithmetic sequence $\{a_i\}$, $a_K = 2001$. If the common difference d is an integer, compute the largest value of d such that $a_{2K} < 2500$.

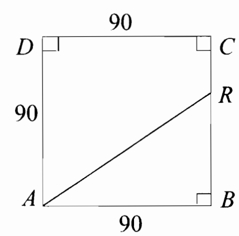
8. You will receive two numbers. Let a be the smaller and b the larger. Define a sequence of numbers as follows: $s_1 = a$, $s_2 = b$, and for $n > 2$, $s_n = as_{n-1}$ if n is odd or bs_{n-1} if n is even. Compute the smallest value of k such that s_k is divisible by 10^{2001}.

15. A three-digit number ABC is called *primal* if A, B, and C are primes and are not all equal. Compute the number of positive three-digit primal numbers.

14. Let T = TNYWR and let $K = \frac{T}{3}$. In the arithmetic sequence $\{a_i\}$, $a_K = 2001$. If the common difference d is an integer, find the minimum value of d so that $a_{K+9} > 10{,}000$.

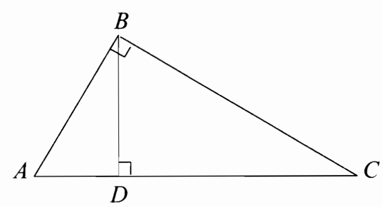
13. Let T = TNYWR and let K be the sum of the digits of T. There are K people in a room, 4 of whom are women. If two people in the room are chosen at random, compute the probability that at least one of them is a woman.

12. Let T = TNYWR. Let $T = \frac{a}{b}$ where a and b are relatively prime. Set $K = a + b$. Base runner R leaves base B running at K feet/second and after x seconds the area of ΔABR is K% of the area of square $ABCD$. If $AB = 90$ feet, compute x.



11. Let T = TNYWR. Compute base b if $\log_b(75T) = 2 + \log_b 3 + \log_b 5$.

10. Let T = TNYWR and let $K = T + 2$. In the diagram, $AD = K^2$ and $DC = 96$. Compute AB.



9. Let T = TNYWR. Let $f(x) + f(x+1) + f(x+2) + f(x+3) = T$ where f is a function defined on the set of integers. If $f(222) = 60$, $f(1776) = 28$, and $f(1999) = -83$, compute $f(2001)$.

8. You will receive two numbers. Let a be the smaller and b the larger. Define a sequence of numbers as follows: $s_1 = a$, $s_2 = b$, and for $n > 2$, $s_n = as_{n-1}$ if n is odd or bs_{n-1} if n is even. Compute the smallest value of k such that s_k is divisible by 10^{2001}.

1. $\frac{3}{4}$

2. 6

3. $\frac{1}{2}$

4. 1

5. $\frac{4\sqrt{6}}{25}$

6. $2\sqrt{6}$

7. 24

---

15. 60

14. 889

13. $\frac{3}{10}$

12. 1.8 or $\frac{9}{5}$

11. 3

10. 55

9. 50

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8. 2002

1. $3s^2 = \dfrac{6h^2\sqrt{3}}{4} \rightarrow \dfrac{s^2}{h^2} = \dfrac{\sqrt{3}}{2} \rightarrow \dfrac{s^4}{h^4} = \boxed{\dfrac{3}{4}}$.

2. $\log_8\left(2^{24}\right)^T = T\log_8(2^3)^8 = 8T\log_8 8 = 8T$. Since $T = \dfrac{3}{4}$, $8T = \boxed{6}$.

3. The first part of the product equals $10x^7 + 5(T-2N)x^6 + \ldots$. Since the coefficient of the x^6 term is 0, $T-2N = 0$ giving $\dfrac{N}{T} = \boxed{\dfrac{1}{2}}$. The result from #2 is unnecessary.

4. Let the diameters of the three smaller arcs be a, b, and c. Then $P = \dfrac{1}{2}(\pi\cdot 2T + \pi\cdot a + \pi\cdot b + \pi\cdot c) = \dfrac{\pi}{2}(2T + a + b + c) = \dfrac{\pi}{2}(2T + 2T) = 2T\pi$. Since $T = \dfrac{1}{2}$, $P = \pi$ and $\dfrac{P}{\pi} = \boxed{1}$.

5. Area $=$ area $\rightarrow \dfrac{1}{2}(5)(5)\sin\theta = \sqrt{(5+T)(5-T)T\cdot T} \rightarrow \sin\theta = \dfrac{2T}{25}\sqrt{25 - T^2}$.

   Since $T = 1$, $\sin\theta = \boxed{\dfrac{4\sqrt{6}}{25}}$.

6. A line with slope $-K$ passing through $A(0, K)$ will also pass through $B(1, 0)$.

   The area of the triangle $= \dfrac{1}{2}(1)(K) = \dfrac{K}{2}$. Since $K = 4\sqrt{6}$, $\dfrac{K}{2} = \boxed{2\sqrt{6}}$.

7. $T = 2\sqrt{6}$ so $K = 24 - 4 = 20$. If $a_K = 2001$, then $a_{2K} = 2001 + Kd$ and if $a_{2K} < 2500$, then $Kd < 499 \rightarrow d < \dfrac{499}{K}$. Since $K = 20$, $d < 24 + \dfrac{19}{20} \rightarrow d = \boxed{24}$.

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15. The digits could be 2, 3, 5, or 7. There would be $4 \cdot 4 \cdot 4 = 64$ primal numbers if A, B, and C could all be equal, but we reject 222, 333, 555, and 777 obtaining $\boxed{60}$ primal numbers.

14. Since $a_{K+9} = 2001 + 9d > 10{,}000$, then $9d > 7999$, making $d > 888.\bar{8} \rightarrow d = \boxed{889}$. T is unnecessary.

13. $T = 889$, so $K = 8 + 8 + 9 = 25$. The probability that both are men $= \dfrac{K-4}{K} \cdot \dfrac{(K-1)-4}{K-1}$. The probability that at least one is a woman is $1 - \dfrac{(K-4)(K-5)}{K(K-1)} = \dfrac{8K-20}{K(K-1)}$. Since $K = 25$, the answer is $\boxed{\dfrac{3}{10}}$.

12. T is unnecessary. $RB = Kx$, so the area of $\Delta ABR = \dfrac{1}{2} \cdot 90 \cdot Kx = \dfrac{K}{100} \cdot 90^2$. K's cancel so $x = \boxed{1.8}$.

11. $T = 1.8$. $\log_b 75T = \log_b b^2 + \log_b 15 = \log_b 15b^2 \rightarrow 15b^2 = 75T \rightarrow b^2 = 5T = 9 \rightarrow b = \boxed{3}$.

10. $T = 3 \rightarrow K = 5$. Since $AB^2 = AD \cdot DC$, $AB^2 = K^2(K^2 + 96) \rightarrow AB = K\sqrt{K^2 + 96}$. Assuming that AB and K are integers, $K = 2, 5, 10$, or 23, making $AB = 20, 55, 140$, or 575. Since $K = 5$, $AB = \boxed{55}$.

9. Since $f(x) + f(x+1) + f(x+2) + f(x+3) = f(x+1) + f(x+2) + f(x+3) + f(x+4)$, then $f(x) = f(x+4) \rightarrow$ $f(x+1) = f(x+5)$, $f(x+2) = f(x+6)$, $f(x+3) = f(x+7)$, etc. Letting $f(1776) = f(x)$, then $f(222) = f(x+2)$, $f(1999) = f(x+3)$ and $f(2001) = f(x+1)$. Thus, $28 + f(2001) + 60 - 83 = T$.

Since $T = 55$, $f(2001) = \boxed{50}$.

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8. Let $a = 24 =$ and $b = 50$. Since $s_3 = ab$, $s_4 = ab^2$, $s_5 = a^2b^2$, $s_6 = a^2b^3$, we have $s_k = s_{2t+1} = a^tb^t$ or $s_k = s_{2t+2} = a^tb^{t+1}$ for $t \geq 1$. Since $a = 2^3 \cdot 3$ and $b = 2 \cdot 5^2$, $s_{2t+1} = 2^{4t} \cdot 3^t \cdot 5^{2t}$ and $s_{2t+2} = 2^{4t+1} \cdot 3^t \cdot 5^{2t+2}$. We have plenty of 2's so if 10^{2001} is to divide s_k, then 5^{2001} must divide either 5^{2t} or 5^{2t+2}. If k is odd, then $2t \geq 2001 \rightarrow t \geq 1001 \rightarrow k = 2003$. If k is even, then $2t + 2 \geq 2001 \rightarrow$ $t \geq 1000$, so $k = \boxed{2002}$ works.

1. In ΔABC, $AB = AC = 1$, $BC = k$, and $\dfrac{\tan B}{\tan A} = 2001$. Compute k^2.

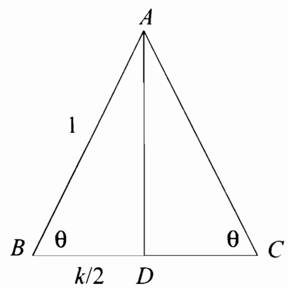
2. A circle intersects the graph of $y = ax^2$ at $x = 1$ or 4 and it is tangent to $y = ax^2$ at a third point. Compute the x-value of the third point.

3. Given $\dfrac{\log(xy)}{\log\left(\dfrac{x}{y}\right)} = \dfrac{1}{2}$, increasing y by 50% decreases x by a factor of k. Compute k.

1. Let $m\angle B = \theta$, then $m\angle A = 180° - 2\theta$ and $\tan A = -\tan 2\theta$. Thus, $\dfrac{\tan B}{\tan A} = \dfrac{\tan\theta}{-\tan 2\theta} = \dfrac{\tan\theta}{\dfrac{2\tan\theta}{\tan^2\theta - 1}}$

$= \dfrac{\tan^2\theta - 1}{2} = 2001 \rightarrow \tan^2\theta = 4003$. Since $AD = \dfrac{\sqrt{4-k^2}}{2}$, then $\tan\theta = \dfrac{\sqrt{4-k^2}}{k}$ and

$\tan^2\theta = \dfrac{4-k^2}{k^2} = \dfrac{4}{k^2} - 1$. Thus, $\dfrac{4}{k^2} - 1 = 4003 \rightarrow k^2 = \boxed{\dfrac{1}{1001}}$.



Alternate solution: Use $\tan A + \tan B + \tan C = (\tan A)(\tan B)(\tan C) \rightarrow \tan A + 2\tan B = (\tan A)(\tan^2 B) \rightarrow$

$1 + \dfrac{2\tan B}{\tan A} = \tan^2 B \rightarrow \dfrac{\tan B}{\tan A} = \dfrac{\tan^2 B - 1}{2}$, and continue as above.

2. Let the circle's equation be $(x-h)^2 + (y-k)^2 = r^2$. Expanding and substituting ax^2 for y gives $a^2x^4 + 0x^3 + (1-2ak)x^2 - 2hx + h^2 + k^2 - r^2 = 0$. This equation's solutions are 1, 4, t, and t since the circle is tangent to the parabola. The coefficient of x^3 equals $-(1+4+2t) = 0$, so $t = \boxed{\dfrac{-5}{2}}$.

3. $\log(xy) = \dfrac{1}{2}\log\left(\dfrac{x}{y}\right) = \log\sqrt{\dfrac{x}{y}} \rightarrow xy = \sqrt{\dfrac{x}{y}} \rightarrow \sqrt{x} = \dfrac{1}{y^{3/2}} \rightarrow x = \dfrac{1}{y^3}$. Increasing y by 50% gives $\dfrac{1}{\left(\dfrac{3}{2}y\right)^3} = \left(\dfrac{8}{27}\cdot\dfrac{1}{y^3}\right) = \dfrac{8}{27}\cdot x$. Thus, $k = \boxed{\dfrac{8}{27}}$.

ARML

2002

THE 27th ANNUAL MEET

In 2002, ARML saw 23 teams take part in Division A and 67 teams in Division B. In addition, Taiwan sent 5 teams and the Philippines sent 1. In Division A, Thomas Jefferson and Massachusetts A matched up on the team and relay rounds, but TJ pulled ahead on the individual and Power Question for a solid victory. In Division B, New York City S trailed Southern California by 8 points after the team round but pulled ahead on the individual and relays for an 8 point victory. The new site at San Jose State proved to be an excellent venue for the second year in a row.

Curt Boddie of New York City received the Samuel Greitzer Distinguished Coach Award for his years as a coach, problem poser and solver, and for his love of fine literature.

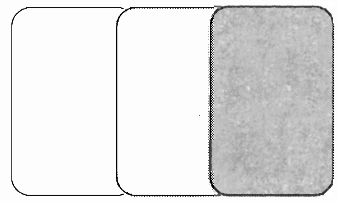
Carrie Kiser Wacker received the Alfred Kalfus Founder's Award for playing a crucial role in establishing the Iowa site. Until recently Carrie was the conference coordinator at Iowa. In a much appreciated effort, she was able to get the university to provide air conditioning.

Austin Shapiro of Northern California received the Zachary Sobol Award for his fine work with his California team.

ARML Team Questions – 2002

T–1. Compute the smallest positive base 10 integer the product of whose digits exceeds 1000.

T–2. 2000 cards of width 2" are placed in a row as suggested in the diagram at the right. The second card overlaps the first card by 1/1000 of an inch and each successive card overlaps the previous card by one-thousandth of an inch <u>more</u>. In inches, compute the length of the row.



T–3. If 240 equals abc, the product of positive integers a, b and c, compute the number of distinct ordered triples (a, b, c) such that a is a multiple of 2, b is a multiple of 3 and c is a multiple of 5.

T–4. $\overleftrightarrow{ADOC}$ is a secant of a circle with center O. A lies outside the circle and D and C lie on the circle. $\overleftrightarrow{ABE}$ is also a secant of circle O and B and E are distinct points lying on circle O. If $AB = BC$, compute, in degrees, the largest possible integer value of the measure of $\angle CAE$.

T–5. Suppose that palindromes with n digits are formed using only the digits 1 and 2 and that each palindrome contains at least one of each digit. Compute the least value of n such that the number of palindromes formed exceeds 2002.

T–6. Let P be a point on side $\overline{ED}$ of regular hexagon $ABCDEF$ such that $EP : PD = 3 : 5$. $\overleftrightarrow{CD}$ meets $\overrightarrow{AB}$ and $\overrightarrow{AP}$ at M and N respectively. Compute $\dfrac{\text{area } \Delta AMN}{\text{area } ABCDEF}$.

T–7. Starting at the origin, a beam of light hits a mirror (in the form of a line) at $A(4, 8)$ and is reflected to the point $B(8, 12)$. Compute the exact slope of the mirror.

T–8. Compute the number of positive integers a for which there exists an integer b, $0 \le b \le 2002$, such that both of the polynomials $x^2 + ax + b$ and $x^2 + ax + b + 1$ have integer roots.

T–9. Shown is a square in which the elements in each row, each column, and each diagonal have the same <u>product</u> N. Let $a, b, c, \ldots, i$ be distinct integers greater than or equal to 1. Compute the least possible value for N.

a	b	c
d	e	f
g	h	i

T–10. Let m be an integer such that $0 < m \le 29$ and let n be a positive integer. There exist rectangles which can be divided into n congruent squares and also into $n + m$ congruent squares. Compute the number of distinct values of m so that for each of those values of m there exists a <u>unique</u> value of n.

1. 2789
2. 2001
3. 10
4. 29 or 29°
5. 21
6. $\dfrac{32}{33}$
7. $\dfrac{1+\sqrt{10}}{3}$
8. 44
9. 216
10. 18

T–1. The largest possible product for a three-digit number is $9 \cdot 9 \cdot 9 = 729$, so the number must have four digits. The thousandth's digit must be at least 2 since 1 doesn't change products obtained from three-digit numbers. Let the number be $2ABC$ where $A \cdot B \cdot C > 500$. If $A = 6$, then $B \cdot C > 500/6 > 83$, an impossibility. If $A = 7$, then $B \cdot C > \dfrac{500}{7} = 71 + \dfrac{3}{7} \rightarrow B = 8, C = 9$. Answer: $\boxed{2789}$.

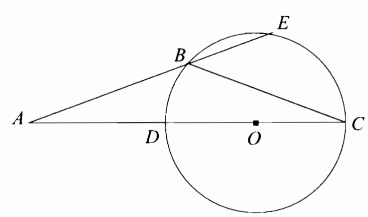
T–2. The right hand edge of the pile is at the following distance from the left hand edge:

$$2 + \left(2 - \frac{1}{1000}\right) + \left(2 - \frac{2}{1000}\right) + \ldots\ldots + \left(2 - \frac{1998}{1000}\right) + \left(2 - \frac{1999}{1000}\right) = 2 \cdot 2000 - \frac{(1 + 2 + \ldots + 1999)}{1000} =$$

$$4000 - \frac{2000 \cdot 1999}{2000} = 4000 - 1999 = \boxed{2001}.$$

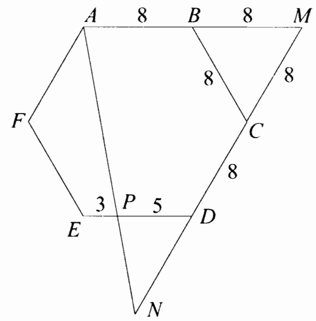
T–3. Since $240 = 2^4 \cdot 3 \cdot 5$ we assign a 2 to a, a 3 to b and a 5 to c, leaving three 2's to assign to a, b, and c in some way. There are 3 ways to assign all three 2's to one letter, 1 way to assign a 2 to each letter, and $3 \cdot 2 = 6$ ways to assign two 2's to one letter and one 2 to another, giving 10 distinct ordered triples (a, b, c) where a, b, and $c > 0$. Ans: $\boxed{10}$.

T–4. Let $m\angle A = x$. Then $m\angle C = x \rightarrow m\overset{\frown}{BD} = 2x$ and by the Exterior Angle Theorem, $m\angle EBC = 2x \rightarrow m\overset{\frown}{EC} = 4x$. Thus, $180° = 2x + m\overset{\frown}{BE} + 4x \rightarrow 6x < 180°$, making $x < 30° \rightarrow m\angle A = \boxed{29°}$.



T–5. We need only consider the first half of the number since the second half will mirror the first half. For an n-digit palindrome there will be $k = \left[\dfrac{n+1}{2}\right]$ entries in the first half. If we could freely chose 1's and 2's there would be 2^k different palindromes, but we're excluding those two with all 1's or all 2's, so there are $2^k - 2 = 2^{[(n+1)/2]} - 2$ palindromes. Hence, $2^{[(n+1)/2]} - 2 > 2002 \rightarrow 2^{[(n+1)/2]} > 2004$.

Thus, $\left[\dfrac{n+1}{2}\right] = 11 \rightarrow n = \boxed{21}$.



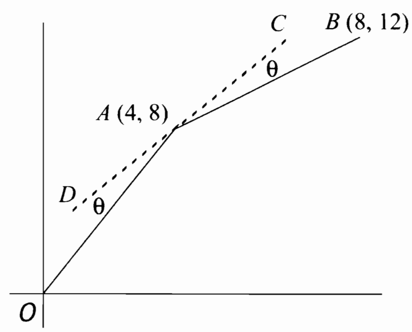
T–6. Let $AB = 8$ and note that ΔMBC is equilateral and $\Delta NAM \sim \Delta NPD$.

Thus, $\frac{ND}{NM} = \frac{PD}{AM} \rightarrow \frac{ND}{ND + 16} = \frac{5}{16} \rightarrow ND = \frac{80}{11}$. The area of

$\Delta NAM = \frac{1}{2} \cdot 16 \cdot \left(16 + \frac{80}{11}\right) \sin \angle M = 8\left(\frac{256}{11}\right)\frac{\sqrt{3}}{2} = \frac{1024\sqrt{3}}{11}$. The

area of $ABCDEF = 6 \cdot \frac{8^2\sqrt{3}}{4} = 96\sqrt{3} \rightarrow \frac{\text{area } \Delta NAM}{\text{area } ABCDEF} =$

$\frac{1024\sqrt{3}\ /\ 11}{96\sqrt{3}} = \boxed{\frac{32}{33}}$.



T–7. Let the equation of the mirror $\overline{DC}$ be $y = mx + b$. The slopes of the mirror, $\overline{OA}$, and $\overline{AB}$ are m, 2, and 1 respectively. Since the angle of incidence equals the angle of reflection,

$\tan\angle CAB = \tan\angle DAO \rightarrow \frac{m - 1}{1 + m} = \frac{2 - m}{1 + 2m} \rightarrow$

$3m^2 - 2m - 3 = 0 \rightarrow m = \boxed{\frac{1 + \sqrt{10}}{3}}$.

T–8. The discriminant of each quadratic must be a perfect square if each is to be factorable over the set of integers. Thus, set $a^2 - 4b = m^2$ and $a^2 - 4(b + 1) = n^2$ for integers m and n. Subtracting cancels a^2 and $4b$ yielding $m^2 - n^2 = (m + n)(m - n) = 4$. Since $m + n$ and $m - n$ are of the same parity, we obtain $m = \pm 2,\ n = 0 \rightarrow m^2 = 4$. Thus, from $a^2 - 4b = m^2$ we obtain $a^2 = 4 + 4b \rightarrow a = \pm 2\sqrt{b + 1}$. Let $a > 0$. For a to be integral, $b + 1$ must be a perfect square, so let $b + 1 = c^2$ for c a positive integer. Then both quadratic expressions are factorable since

$$x^2 + ax + b = x^2 + 2c + c^2 - 1 = (x + c - 1)(x + c + 1)$$

$$x^2 + ax + b + 1 = x^2 + 2c + c^2 = (x + c)^2$$

Thus, for both expressions to be factorable it is necessary and sufficient that $b + 1$ be a perfect square and that $a = 2\sqrt{b + 1}$. For $0 \le b \le 2002$, $b + 1$ is a perfect square for $\left[\sqrt{2002}\right] = 44$ positive values of b. Thus, a can take $\boxed{44}$ values.

T–9. Note that $N^4 = (abc)(def)(ghi)N = (aei)(beh)(ceg)(def) = (ai)(bh)(cg)(def)e^3 = (abc)(def)(ghi)e^3$. Thus, $N = e^3$. We seek the least positive integer e such that $abc = e^3$ for distinct integers a, b, c, and e. Letting one of a, b, or c equal 1, say $b = 1$, then $ac = e^3$. Now, e can't be prime since if it were then $a = 1$ and $c = e^3$ or $a = e$ and $c = e^2$, both of which are impossible since a, b, and c must be distinct. Nor can $e = 4$ since then $a = 1$ and $c = 4$ or $a = c = 2$. Thus, the least value for e is 6 making $N = 6^3 = \boxed{216}$.

Note: Such a square exists. Let $b = 1$, making $h = 2^2 \cdot 3^2$. Then $g = 2$ and $i = 3$ or vice-versa. Setting $g = 2$, then $c = 2 \cdot 3^2$ and the rest of the entries are determined in similar fashion, giving the figure at the right.

$2^2 \cdot 3$	1	$2 \cdot 3^2$
3^2	$2 \cdot 3$	2^2
2	$2^2 \cdot 3^2$	3

T–10. Without loss of generality, consider a rectangle $ABCD$ that can be divided into $n + m$ squares of side 1 and n larger squares of side x. Since the dimensions of $ABCD$ are integral, x must be a rational number. Let $x = \frac{p}{q}$ where p and q are relatively prime. Since $x > 1$, then $p > q$. Since the area of

$ABCD = (n+m)\cdot 1 = n\left(\frac{p}{q}\right)^2$, we solve for n to obtain $n = \dfrac{mq^2}{p^2 - q^2} = \dfrac{mq^2}{(p-q)(p+q)}$. Since p and q are relatively prime, neither $p - q$ nor $p + q$ divides q^2 so both must divide m and they must have the same parity. If m has two odd factors, say $m = ijk$ where j and k are odd, we have

$p - q = j, p + q = k$ or $p - q = 1, p + q = jk$ giving two different values for n, namely $\dfrac{i(j-k)^2}{4}$ or $\dfrac{i(jk-1)^2}{4}$ respectively. So m must have only 1 odd factor. If m is to be odd, it must be an odd prime. There are 9 of those less than or equal to 29, namely, 3, 5, 7, 11, 13, 17, 19, 23, and 29. If r is an odd prime, the factors of $2r$ or $4r$ with the same parity are $(1, r)$ and $(2, 2r)$ and they both generate the same value for n, namely $\dfrac{(r-1)^2}{4}$. This is an integer since $r - 1$ is even. Thus $m = 2r = 6, 10, 14, 22, 26$ and $m = 4r = 12, 20$, and 28 generate an additional 8 values for m. If $m = 8r$, then $(2, 4r)$ and $(4, 2r)$ have the same parity and $p - q = 2, p + q = 4r$ gives $n = (2r-1)^2$ while $p - q = 4, p + q = 2r$ gives $n = (r-2)^2$. These are equal for $r = 1$, so for $m = 8$ there is a unique value for n. For $m = 1, 2$ or 4 there are no values for n since both $p - q = 1, p + q = 1$ and $p - q = 2, p + q = 2$ yield $q = 0$. Hence, the total number of values is $9 + 8 + 1 = \boxed{18}$.

In all these problems a *clique* is a non-empty set of students and students may be members of more than one clique. Also, lowercase letters denote students; uppercase letters denote cliques. For example, $C = abcd$ is a clique of four students, a, b, c, and d. We represent a collection of cliques with set notation, e.g. $\{ abcd, efgh \}$ is a collection of two cliques of four students each. We use S to stand for the set of all students, i.e. the student body. There is a finite number of students.

At Archimedes Academy, the faculty is concerned about students' tendency to form cliques and it hires an anthropologist to study the cliques. The anthropologist finds that the cliques at AA satisfy the following three conditions:

A1. For any two students, there is exactly one clique of which they are both members.

A2. If a student a is not a member of a clique C, then there exists exactly one clique D of which a is a member and that has no members in common with C.

A3. There are three students that are not all members of the same clique.

1 a) If $S = \{a, b, c, d\}$, determine a collection of cliques that satisfies A1–A3.

b) If $S = \{a, b, c\}$, prove that there is no collection of cliques that satisfies A1–A3.

2. If $S = \{a, b, c, d, e, f, g, h, i\}$, determine a collection of cliques that satisfies A1–A3.

In problems #3 – #6 assume that C, D, and E are cliques at Archimedes Academy where the collection of cliques satisfies A1–A3 above.

<u>A clique D will be called *exclusive* of clique C if either (i) $C = D$ or (ii) $C \cap D = \varnothing$.</u>

3. Prove that if C is exclusive of D, and D is exclusive of E, then C is exclusive of E.

<u>Let $|C|$ denote the number of students in clique C.</u>

4. a) Prove that if C and D are exclusive, then $|C| = |D|$.

b) Prove that if C and D are not exclusive, then $|C| = |D|$.

Let $[a]$ denote the set of all cliques of which a is a member, let $[C]$ denote the set of all cliques exclusive of C and let $\left|[a]\right|$ and $\left|[C]\right|$ be the number of elements in the specified set. Thus, $[a] = \{C : a \in C\}$ and $[C] = \{D : D = C \text{ or } D \cap C = \varnothing\}$.

5. Prove that $\left|[C]\right| = |C|$.

6. Prove that if C and D are distinct cliques, then $\left|[C]\right| = \left|[D]\right|$.

7. Prove that the number of students at Archimedes Academy must be a perfect square.

Faculty at Hausdorff High were similarly concerned and called in the same anthropologist. At HH the anthropologist found that the cliques satisfy the following four conditions:

H1. For any two students, there is exactly one clique of which both are members.
H2. For any two cliques, there is exactly one student who is member of both.
H3. There exist three students who are not all members of the same clique.
H4. Every clique has at least three students.

8. a) Prove that if the student body has exactly four students, there is no collection of cliques satisfying H1–H4.

   b) Prove that it is impossible for H1–H4 to be satisfied by a student body of fewer than 6 students.

9. Find one collection of students and cliques satisfying H1–H4 in which each clique has 3 members.

10. Is it possible for a collection of students and cliques to satisfy H1–H3 but not H4? Justify your answer.

11. Let S be a set of students satisfying H1–H4, and let C be a clique in S. Let S_C be the set of students in S and not in C, i.e. $S_C = \{s \in S : s \notin C\}$. Cliques in S_C are formed by removing the members of C from the cliques in S. Show that S_C satisfies the Archimedian conditions A1–A3.

12. If $|C| = n$ for some clique C, find <u>with proof</u> the total number of students at Hausdorff High in terms of n.

1. a) Let $T = \{ab, ac, ad, bc, bd, cd\}$. Then T is a collection of cliques satisfying A1–A3.

We can show (not necessary for credit) that no clique can have more than two members. First, by A3 no clique can have four members. Assume that *abc* is a clique. Then *da, db*, and *dc* must all be distinct cliques by A1 (since *a*, *b*, *c*, cannot be members of any other single clique). But by A2, if *da* is a clique, *bc* must also be a clique, which violates A1.

Since all cliques have two members, we can represent them as <u>rows</u> and <u>columns</u> but not diagonals in an array:

a	*b*
c	*d*

The array yields *ab*, *ac*, *bd*, *cd* as cliques. So far this is not sufficient because we know that *a* and *d* are also members of a clique. So we arrange *a*, *b*, *c*, *d* into a second array incorporating *ad* as a row, also yielding *bc* as a clique.

a	*d*
b	*c*

b) Let *ab* be a clique. By A2, *c* must be a member of a clique that does not include *a* or *b*. Therefore, *c* must be a singleton clique. However, *bc* is also a clique by axiom A1, so *a* must be a singleton clique by axiom A2 applied to *bc*. This is a contradiction because *a* is now a member of two cliques that have no members in common with clique *c*, namely, the clique *a*, and the clique *ab*.

2. Note that two-element cliques no longer work: if *ab* is a clique and *cd*, *ce* are both cliques, then A2 is violated since *c* is a member of two cliques disjoint from *ab*. We will look for a collection of cliques each of which has three elements. Arrange the letters *a-i* in a 3×3 array alphabetically:

a	*b*	*c*
d	*e*	*f*
g	*h*	*i*

Using columns and rows but not diagonals, this arrangement yields cliques *abc*, *def*, *ghi*, *adg*, *beh*, *cfi*. Now *a* and *e* also share a clique, so we need to start building a second array with *ae* as part of a row. To avoid duplicating the *abc* clique, we arrange *b* and *c* along the main diagonal, and similarly, *d* and *g* outside any row or column with *a*. By A1, *b* cannot be in the same row or column as *e* since we already have a clique with *be* in the first array. In the array below these restrictions place *b* in the lower right hand corner.

a	*e*	
	c	*d*
	g	*b*

Now the choices for the other three spots are severely restricted: by axiom A1, the upper-right corner cannot be *f* (already in *def*) or *h* (already in *beh*) so it must be *i*. The lower-left corner must be *f* and the middle left must be *h*:

a	*e*	*i*
h	*c*	*d*
f	*g*	*b*

Notice that the four cliques containing *a* exhaust all other members of *S*, so we're done. Thus, a collection of cliques satisfying A1–A3 is $\{abc, def, ghi, adg, beh, cfi, aei, hcd, fgb, ahf, ecg, idb\}$.

Note: Should we consider collections of cliques in which some cliques have a different number of members from others? In 4a and 4b we prove $|C| = |D|$, so the answer is no.

3. If $C = D$, then we are done, since $D = E \Rightarrow C = E$ and $D \cap E = \varnothing \Rightarrow C \cap E = \varnothing$, so in either case C is exclusive of E. If $C \cap D = \varnothing$ and $D = E$, then $C \cap E = \varnothing$ and so C is exclusive of E. The final possibility is that $C \cap D = \varnothing$ and $D \cap E = \varnothing$. Assume that $C \cap E \neq \varnothing$. Then there exists some $x \in (C \cap E)$. Now $x \notin D$ by the hypothesis that $C \cap D = \varnothing$. But then by axiom A2, C and E cannot be distinct cliques. So $C = E$ and so C is exclusive of E.

4. a) Trivial if $C = D$, so assume $C \cap D = \varnothing$. Let the students in clique C be $c_1, c_2, c_3, \ldots$ and the students in clique D be $d_1, d_2, d_3, \ldots$. Then c_1 and d_1 belong to some clique, call it c_1d_1. By axiom A2, c_2 belongs to another clique that is exclusive of c_1d_1. That clique is not exclusive of D by axiom A2

(or else there would be two cliques containing c_2 exclusive of *D*) so it contains some member of *D*, say d_2. We can continue constructing cliques exclusive of c_1d_1 until we exhaust the members of *C*; each such clique contains a distinct member of *D* (again by A2). So $|C| \le |D|$. The same process applied to *D* shows that $|D| \le |C|$. So $|C| = |D|$. (Technically, we've used one-to-one functions to map *C* into *D* and then *D* into *C*, which, in the case of infinite cliques would require the Schröder-Bernstein theorem to infer that $|C| = |D|$.)

b) Let the students in clique *C* be $a, c_1, c_2, \ldots$ and the students in clique *D* be $a, d_1, d_2, \ldots$. We use the same construction as in 3a, beginning with the clique c_1d_1 and choosing cliques containing $c_2, c_3, \ldots$ each exclusive of c_1d_1. This time the fact that the cliques are not exclusive of *D* is guaranteed by problem 3: if c_1d_1 is exclusive of some clique containing c_2 and that clique were exclusive of *D*, then c_1d_1 would be exclusive of *D*, which it isn't since it contains d_1.

5. Let the students in clique *C* be $c_1, c_2, \ldots, c_n$ and let $d \notin C$. Let *D* be a clique containing c_1, d. By problem 4, *D* has *n* members. Notice that by construction *D* is not exclusive of *C*. Now by A2, for each $s \in D$, there is a clique C_s that contains *s* and is exclusive of clique *C*. We'll show that the cliques C_s actually form a complete collection of all the cliques in $[C]$. If there were a clique C' in $[C]$, then C' must intersect *D*. If not, then C' and *C* are exclusive and so are C' and *D* which would make *C* and *D* exclusive, but they can't be exclusive since they intersect. That means that every C' in $[C]$ intersects with *D* for one value *s*; it can't intersect in two or more values by A1. Moreover, by A2, it is true that each *s* in *D* but not in *C* is a member of only one clique that does not intersect *C*. Therefore, the collection of cliques that are exclusive of *C* consist of *C* itself and one class C_s for which every *s* which is in *D* but not in *C*. There are $|D|$ possible cliques that are exclusive of *C*, so $|[C]| = |D|$. Since *D* and *C* intersect, $|D| = |C|$. So $|[C]| = |D| = |C|$.

6. By problem 5, $|[C]| = |C|$ and $|[D]| = |D|$. By problem 4, $|C| = |D|$. Thus, $|[C]| = |[D]|$ by the transitive property.

7. Pick some clique C. By A2, every student at Archimedes Academy belongs to a clique in $[C]$, either C itself or some clique C_i exclusive of C. But by the uniqueness provision of A2, cliques C_i are disjoint. So $|S| = \sum_{\text{those } C_i \in [C]} |C_i| = \sum_{\text{those } C_i \in [C]} |C|$ since $|C_i| = |C|$ by problem 4. By problem 5, there are exactly $|C|$ summands, so $|S| = |C|^2$.

8. a) Call the students a, b, c, d. Then by H1 and H4, ab is a clique with another member. Assume without loss of generality that this member is c; by H3 that clique cannot contain d. Now by H1, a and d are members of a clique ad which contains another member. But that contradicts H1, since both b and c are already members of another clique with a.

b) Start with a clique abc by H4. There is at least one other student d by H3. Now ad is a clique with another member, necessarily neither b nor c, so call it e. Then bd is a clique with another member, necessarily neither a, c, nor e, so call it f. So at the very least we need 6 students.

9. The axioms are satisfied by the cliques *abc*, *ade*, *afg*, *bdf*, *beg*, *cdg*, *cef* for $S = \{a, b, c, d, e, f, g\}$.

10. Using $S = \{a, b, c\}$, the cliques ab, ac, and bc satisfy H1–H3. Clearly every two students are in exactly one clique, every two cliques share a student, and there exist three students who are not members of the same clique.

11. A1. For any two students, there is exactly one clique of which they are both members.

Suppose that $a, b \in S_C$ and there are distinct cliques C_1 and C_2 in S_C containing both a and b. Then $C_1 \subseteq C_1'$ where C_1' is a clique in S and $C_2 \subseteq C_2'$ where C_2' is a clique in S. But then C_1' and C_2' violate H1.

A2. If a student s is not a member of a clique C, then there exists exactly one clique D of which s is a member and which has no members in common with C.

Existence: Choose a clique D in S_C and a student $s \notin D$. We need to find a clique E containing s as a member with $E \cap D = \varnothing$. Now for some clique D' in S, $D \subseteq D'$. By H2, $D' \cap C$ contains exactly one student, call it t. Then by H1 there exists a clique E' in S containing s and t and by H2 that clique contains no other member of D. Therefore, let E be the clique in S_C that results from removing t from E'.

Uniqueness: Guaranteed by H1 since in the previous construction E' was unique.

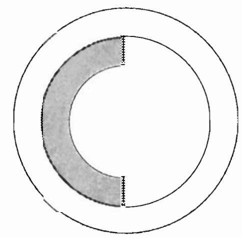
A3. There are three students that are not all members of the same clique.

We show how to find three such students. If $a \in S_C$, then for some $c \in C$, ac is a clique distinct from C (by H1 and the fact that $a \notin C$), so by H4 there is a third student $b \in ac$. But then $b \notin C$ by H2 applied to C and ac, so $b \in S_C$. Now C has at least three members so pick $c' \neq c$ in C and repeat the process to find a student $d \in ac'$. But $d \neq b$ because ac and ac' are distinct cliques, so by H2 they have only one student in common in S, hence only one in common in S_C. So a, b, d are distinct students in S_C. Since ab and ad are distinct cliques in S, a, b, d cannot all be members of any other single clique in S (by H2) and so are not members of a single clique in S_C.

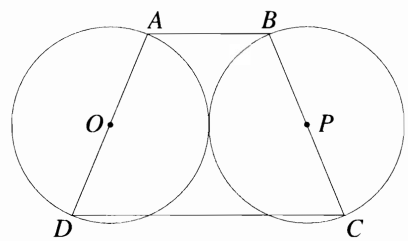
12. Since $|S_C| = |S| - |C|$ we have $|S| = |S_C| + |C|$. Since S_C satisfies A1–A3, by problem 8, then $|S_C| = |D|^2$, where D is any clique in S_C. But $|D| = n - 1$, because if we consider the extension D' of D to S, $|D' \cap C| = 1$ by axiom H2. So $|S| = (n-1)^2 + n = n^2 - n + 1$. Incidentally, this shows that the smallest student body is given by $3^2 - 3 + 1 = 7$ students, and that the next smallest is $4^2 - 4 + 1 = 13$ students.

I–1. For $A > B$, if $\underline{ABA}_{10} - \underline{BAB}_{10}$ is divisible by exactly three distinct primes, compute the largest possible value of B.

I–2. Starting with the outermost ring, a farmer cuts a circular field in rings of uniform width. As an example, the diagram shows one and a half trips around the field. If the width of each ring is 5 feet and if after the farmer has made eight and a half trips around the field, he has cut half the field, compute the radius of the field in feet.



I–3. The legs of isosceles trapezoid $ABCD$ are diameters of tangent circles O and P. If $AB = 90$ and $CD = 1000$, compute the height of the trapezoid.



I–4. Compute the number of ordered pairs of integers (x, y) with $1 \le x < y \le 100$ such that $i^x + i^y$ is a real number. Note: $i = \sqrt{-1}$.

I–5. Let k be a positive integer. The intersection of the graphs of $y < k$ and $y > |x|$ contains at least 90 lattice points. Compute the smallest value of k.

I–6. Let a be the integer such that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{22} + \frac{1}{23} = \frac{a}{23!}$. Compute the remainder when a is divided by 13.

I–7. If $f(x) = (x - 3)^2 - 1$, compute the set of real numbers such that $f(|x|) = |f(x)|$.

I–8. $ABCDEFG$ is a regular heptagon of side 1001. Compute $AC^2 - (DE)(FB)$.

1. 7

2. 145

3. 300

4. 1850

5. 11

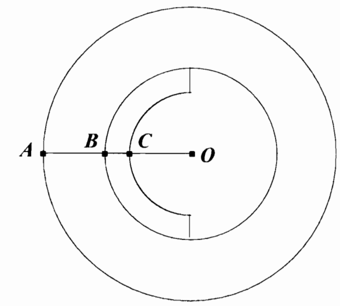
6. 7

7. $0 \le x \le 2$ or $4 \le x$. Alternately, $\left[\,0,2\,\right] \cup \left[\,4,\infty\,\right)$.

8. 1002001

I–1. $ABA_{10} - BAB_{10} = (101A + 10B) - (101B - 10A) = 91(A - B) = 7 \cdot 13(A - B)$. To maximize B, choose A as large as possible and minimize $A - B$. If $A = 9$, then for $B = 7$, $A - B = 2$, a prime. Thus, $B = \boxed{7}$.

I–2. Let $OA = r$, $OB = r - 40$, and $OC = r - 45$. Then the area of the large ring plus half the area of the small ring equals half the area of a circle.

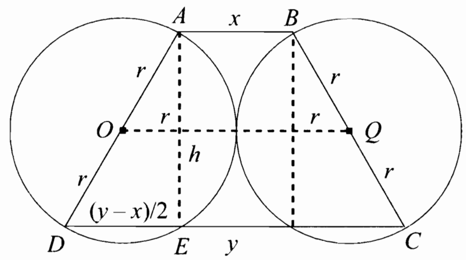


$$\pi r^2 - \pi(r-40)^2 + \frac{1}{2}\left(\pi(r-40)^2 - \pi(r-45)^2\right) = \frac{1}{2}\pi r^2 \rightarrow$$

$$r^2 - (r^2 - 80r + 1600) + \frac{1}{2}(10r - 425) = \frac{r^2}{2} \rightarrow$$

$r^2 - 170r + 3625 = 0 \rightarrow (r - 25)(r - 145) = 0$. Thus, $r = 25$ or 145, but 25 is too small, so $r = \boxed{145}$.

I–3. Let $AB = x$ and $DC = y$. Then $DE = \dfrac{y - x}{2}$. The median of trapezoid $ABCD$ equals $OQ = \dfrac{x + y}{2}$, but that also equals $2r$ which equals AD and BC. Thus, $AD = \dfrac{x + y}{2}$, giving



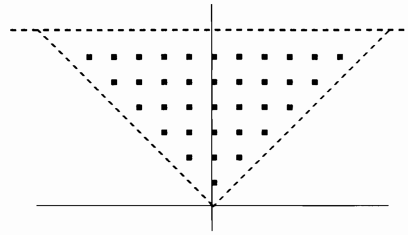
$\left(\dfrac{y-x}{2}\right)^2 + h^2 = \left(\dfrac{y+x}{2}\right)^2 \rightarrow h^2 = \dfrac{1}{4}(4xy) = xy$. Thus, $h = \sqrt{xy}$ and since $x = 90$ and $y = 1000$,

$h = \sqrt{90000} = \boxed{300}$. Note that the area of the trapezoid equals $\sqrt{xy}\left(\dfrac{x + y}{2}\right)$, the product of the arithmetic and geometric means of the bases.

I–4. $i^x + i^y$ is real when both x and y are even. There are 10,000 ordered pairs of (x, y) and 2500 of these will have x and y both even. In $2500 - 50 = 2450$ pairs we have $x \neq y$ and in $(1/2)(2450) = 1225$ pairs we have $x < y$. Since $i^1 + i^3 = 0$, but $i^1 + i^5 = 2i$, it is clear that $i^x + i^y$ is also real when x and y are both odd as long as their difference is 2 more than a multiple of 4. There are 2500 ordered pairs with both x and y odd and $(1/2)(2500) = 1250$ pairs with $x \leq y$. In half of those $y - x = 0 \pmod 4$ and in the other half $y - x = 2 \pmod 4$. Thus, there are $(1/2)(1250) = 625$ pairs where x and y are odd.

Answer: $1225 + 625 = \boxed{1850}$.

I–5. If we intersect the region with vertical lines drawn through lattice points we see that from left to right the number of points will be $1 + 2 + \ldots + (k-2) + (k-1) + (k-2) + \ldots + 2 + 1 =$ $2\left(\dfrac{1+(k-2)}{2}\right)(k-2) + (k-1) = (k-1)^2$. Or, use horizontal



lines to obtain $1 + 3 + \ldots + (2k-3) = (k-1)^2$. For $(k-1)^2 > 90$, $k = \boxed{11}$.

I–6. Since $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{22} + \frac{1}{23} = \dfrac{23!(1) + 23!\left(\frac{1}{2}\right) + 23!\left(\frac{1}{3}\right) + \cdots + 23!\left(\frac{1}{22}\right) + 23!\left(\frac{1}{23}\right)}{23!}$, then

$a = 23!(1) + 23!\left(\frac{1}{2}\right) + 23!\left(\frac{1}{3}\right) + \cdots + 23!\left(\frac{1}{22}\right) + 23!\left(\frac{1}{23}\right)$. Every term in this expression is divisible by 13 except $23!\left(\frac{1}{13}\right)$. Consequently, the remainder when a is divided by 13 is the remainder when $\dfrac{23!}{13}$ is divided by 13. Since the remainders when 14, 15, 16, . . . , 23 are divided by 13 are the same, respectively, as the remainders when 1, 2, 3, . . . , 10 are divided by 13, the remainders R of the following are the same when divided by 13: $(1 \cdot 2 \cdot 3 \cdot \ldots \cdot 12)(14 \cdot 15 \cdot 16 \cdot \ldots \cdot 23)$, $(1 \cdot 2 \cdot 3 \cdot \ldots \cdot 9 \cdot 10)^2(11 \cdot 12)$, and $((2 \cdot 7) \cdot (3 \cdot 9) \cdot (4 \cdot 10) \cdot (5 \cdot 8) \cdot 6)^2(11 \cdot 12)$. Note that we have paired up numbers whose product is 1 more than a multiple of 13. This means that the remainder when a is divided by 13 is given by the remainder when $6^2 \cdot 11 \cdot 12$ is divided by 13. Since $10 \cdot 11 \cdot 12 = 1320 = 13 \cdot 100 + 20$ gives the same remainder, we need only look at the remainder when 20 is divided by 13. Answer: $\boxed{7}$.

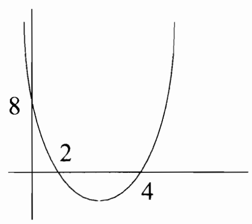
<u>Alternate solution</u>:

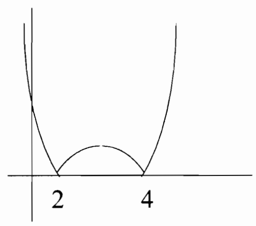
$R = (1 \cdot 2 \cdot 3 \cdot \ldots \cdot 12)(14 \cdot 15 \cdot 16 \cdot \ldots \cdot 23) \pmod{13} \equiv 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 12 \cdot 1 \cdot 2 \cdot \ldots \cdot 10 \pmod{13} \equiv$

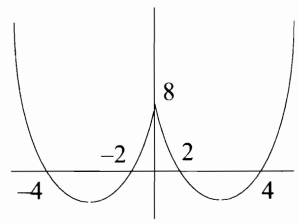
$(12!)(10!) \pmod{13}$. By Wilson's Theorem, $(p-1)! \equiv -1 \pmod{p}$, giving $R \equiv (-1)(10!) \pmod{13}$.

Since $10! \cdot 11 \cdot 12 = 12! \equiv -1 \pmod{13}$, then $(10!)(-2)(-1) \equiv (-1) \pmod{13}$. Thus, $-2R \equiv -1 \pmod{13}$ so $2R \equiv 1 \pmod{13}$, making $R = 7$.

I–7. The leftmost diagram is the graph of $f(x) = (x-3)^2 - 1$, the middle is the graph of $|f(x)| = |(x-3)^2 - 1|$, a selective reflection over the x-axis, and the rightmost is the graph of $f(|x|) = \left((|x|-3)^2 - 1\right)$, a reflection over the y-axis.



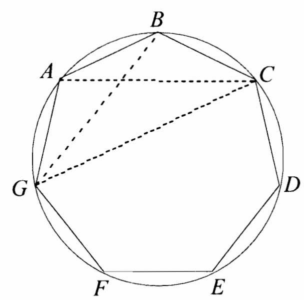




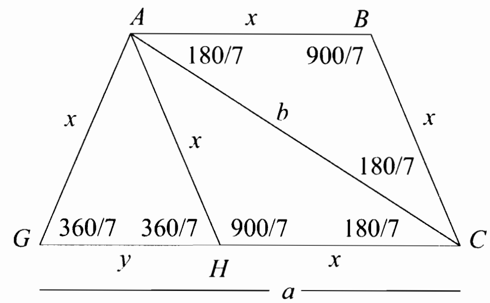
The solution set consists of the x-values for which the middle and right hand graphs overlap, namely, $\boxed{0 \le x \le 2 \text{ or } 4 \le x}$.

I–8. In all these solutions note that $AB = BC = \ldots = GA = x$, $GB = AC = b$, and $FB = GC = a$. Thus, $AC^2 - (DE)(FB) = AC^2 - (AB)(GC)$.

Since $ABCG$ is cyclic, we have by Ptolemy's Theorem, $AC \cdot GB = AB \cdot GC + AG \cdot BC$. The trapezoid is isosceles so $AC = GB$ giving $AC^2 = AB \cdot GC + AB^2 \rightarrow AC^2 - AB \cdot GC = AB^2 = 1001^2 = \boxed{1002001}$.



Note: this method would establish that for any regular n–gon $A_1A_2A_3 \ldots A_n$, the following relationship holds: $(A_1A_3)^2 - (A_1A_2)(A_nA_3) = (A_1A_2)^2$.



Alternate solution using Law of Cosines: Draw $\overline{AH} \parallel \overline{BC}$ making rhombus $ABCH$. Let $GH = y$. Since $ABCG$ is an isosceles trapezoid and each interior angle of the regular heptagon equals 900/7 in degrees, the rest of the angles are as marked.

Using ΔACH: $AC^2 = b^2 = x^2 + x^2 - 2(x)(x)\cos\frac{900}{7}$. Since $\frac{900}{7} + \frac{360}{7} = 180$, then $\cos\frac{900}{7} = -\cos\frac{360}{7}$, giving $b^2 = 2x^2 + 2x^2\cos\frac{360}{7}$.

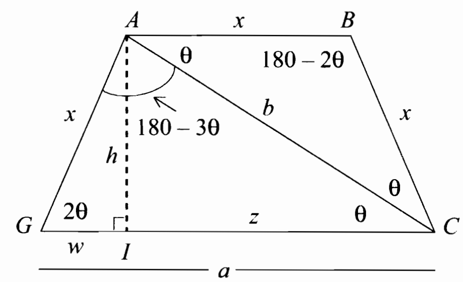
Using ΔAHG: $AG^2 = x^2 = x^2 + y^2 - 2(x)(y)\cos\frac{360}{7}$. Thus, $0 = y^2 - 2xy\cos\frac{360}{7} \rightarrow$

$y = 2x\cos\frac{360}{7} \rightarrow xy = 2x^2\cos\frac{360}{7}$.

$$AC^2 - (GC)(AB) = b^2 - (x+y)x = b^2 - x^2 - xy = \left(2x^2 + 2x^2\cos\frac{360}{7}\right) - x^2 - \left(2x^2\cos\frac{360}{7}\right) = x^2.$$

Since $x = 1001$, the answer is 1002001.

Alternate solutions using the Law of Sines:



Using ΔABC, $\frac{b}{\sin(180-2\theta)} = \frac{x}{\sin\theta}$ and since $\sin(180-2\theta) = \sin 2\theta$, $b = \frac{x(2\sin\theta\cos\theta)}{\sin\theta} = 2x\cos\theta$.

Using ΔAIC, $\cos\theta = \frac{z}{b} \rightarrow z = b\cos\theta$, and using ΔAGI

$\cos 2\theta = \frac{w}{x} \rightarrow w = x\cos 2\theta$. Since $a = w + z$, $a = x\cos 2\theta + b\cos\theta = x\cos 2\theta + (2x\cos\theta)\cos\theta$.

Thus, $b^2 - ax = 4x^2\cos^2\theta - \left(x\left(2\cos^2\theta - 1\right) + 2x\cos^2\theta\right)x = x^2$.

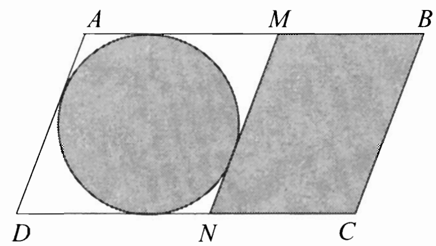
Or, one could use $b = 2x\cos\theta$ and from ΔAGC obtain $\frac{a}{\sin 3\theta} = \frac{x}{\sin\theta} \rightarrow a = \frac{x\left(3\sin\theta - 4\sin^3\theta\right)}{\sin\theta} =$

$3x - 4x\sin^2\theta = 3x - 4x(1-\cos^2\theta)$ and continue as above.

ARML Relay #1 – 2002

R1–1. Suppose that in base 10, ELEVEN stands for a number divisible by 11. If different letters stand for different digits, compute the largest value for ELEVEN.

R1–2. Let T = the sum of the digits of TNYWR. In the diagram, $ABCD$ is a parallelogram, the circle is inscribed in rhombus $AMND$, and the areas of the shaded regions are equal. If $NC = T - 41$, compute the circumference of the circle.

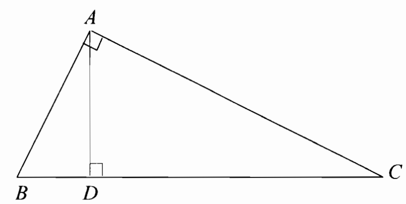


R1–3. Let T = TNYWR. Solve for x: $\dfrac{1+\sqrt{x}}{1-\sqrt{x}} = T$.

ARML Relay #2 – 2002

R2–1. Ed spends \$1.20 every week on noodles. The price of a package of noodles changes each week, cycling through 10, 15, and 20 cents in a regular fashion. To the nearest penny, Ed spends N cents on average per package of noodles. Compute N as an integer without any dollar or cents notation.

R2–2. Let T = TNYWR. In ΔABC, $\overline{AB} \perp \overline{AC}$, $\overline{AD} \perp \overline{BC}$, the area of $\Delta ABD = 2$, and the area of $\Delta ADC = T$. Compute $\tan^2 \angle B$.



R2–3. Let T = TNYWR. Compute the area of the smallest regular hexagon containing the points

$A\left(\log \dfrac{7}{100T}, 0\right)$ and $B\left(\log \dfrac{100T}{7}, 0\right)$.

Relay #1:

R1–1. 989791

R1–2. 8

R1–3. $\dfrac{49}{81}$

Relay #2:

R2–1. 14

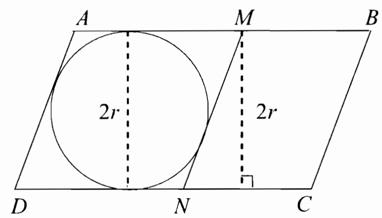
R2–2. 7

R2–3. $6\sqrt{3}$

Solutions to ARML Relay #1 – 2002

R1–1. If ELEVEN is divisible by 11, then E – L + E – V + E – N = 3E – (L + V + N) is divisible by 11. Set E = 9 → 27 – (L + V + N) is divisible by 11. ELEVEN is largest when L + V + N = 16 and when L = 8, V = 7, and N = 1. Thus, ELEVEN = $\boxed{989791}$.

R1–2. $T = 9 + 8 + 9 + 7 + 9 + 1 = 43$, making $NC = 43 - 41 = 2$. Since $\pi(r)^2 = 2r(NC)$, then $\pi r = 2NC$, giving $2\pi r = 4NC$. Thus, the circumference of the circle is $4 \cdot 2 = \boxed{8}$.



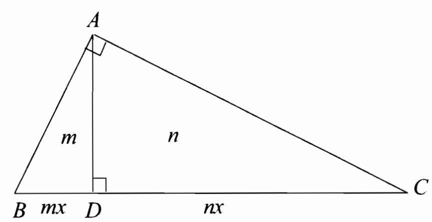
R1–3. $T = 8$. Since $1 + \sqrt{x} = T - T\sqrt{x}$, then $T\sqrt{x} + \sqrt{x} = T - 1 \rightarrow \sqrt{x} = \dfrac{T-1}{T+1} \rightarrow x = \left(\dfrac{T-1}{T+1}\right)^2$.

Hence $x = \boxed{\dfrac{49}{81}}$.

Solutions to ARML Relay #2 – 2002

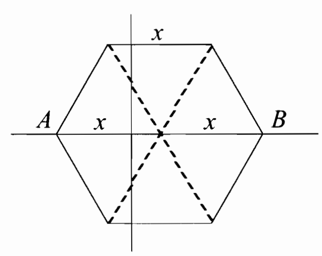
R2–1. Every three weeks Ed buys 12 packages at \$.10/package, 8 packages at \$.15/package, and 6 packages at \$.20/package, making a total of 26 packages costing \$3.60. Since 360/26 = 13.8, Ed spends an average of \$.14 per package. Pass back $\boxed{14}$.

R2–2. $T = 14$. In general, if the areas of ΔABD and ΔADC are m and n respectively, then their bases $\overline{BD}$ and $\overline{DC}$ have lengths mx and nx respectively, since the triangles share an altitude. Since AD is the geometric mean between BD and DC, $AD = \sqrt{(mx)(nx)} = x\sqrt{mn}$. Thus,



$\tan B = \dfrac{x\sqrt{mn}}{mx} = \dfrac{\sqrt{n}}{\sqrt{m}}$ and $\tan^2 B = \dfrac{n}{m}$. Since $m = 2$ and $n = T = 14$, $\tan^2 B = \dfrac{14}{2} = \boxed{7}$.

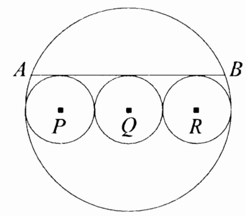
R2–3. $T = 7$. The smallest regular hexagon will have A and B at endpoints of the longest diagonal whose length will be $\log\dfrac{100T}{7} - \log\dfrac{7}{100T} =$ $\log\dfrac{10^4 T^2}{49} = \log\dfrac{10^4 7^2}{49} = 4$. Since AB is twice the side x, the side of the hexagon is 2 and the hexagon's area is $6\left(\dfrac{2^2\sqrt{3}}{4}\right) = \boxed{6\sqrt{3}}$.



Note: Pass answers from position 1 to 8 and from position 15 to 8.

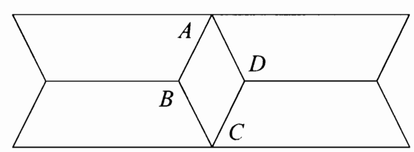
1. Given that $2002 = 2 \cdot 7 \cdot 11 \cdot 13$, compute the number of positive two-digit factors of 2002.

2. Let T = TNYWR and let $K = T + 2$. The large circle has a radius of K. The congruent circles centered at P, Q, and R are tangent to exactly two other circles and circle Q is concentric with the large circle. Points A and B lie on the large circle so that $\overline{AB}$ is tangent to circles P, Q, and R. Compute AB.



3. Let T = TNYWR. If $|a + bi| = T$ and $|a + 2bi| = \sqrt{T^2 + 96}$, compute $|a|$.

4. Let T = TNYWR. Four congruent isosceles trapezoids are arranged as shown. The length of the longer base of each trapezoid is T more than the shorter base. If the area of quadrilateral $ABCD$ is the product $(20.02)T$, compute the height of a trapezoid.



5. Let T = TNYWR. Let K be the integer closest to T. A bag contains K marbles, half of which are green; the other half are red. Two marbles are drawn randomly and without replacement from the bag. Compute the probability that their colors differ.

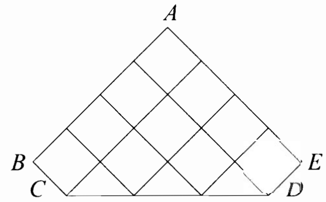
6. The number you will receive should be a fraction. Let $\frac{N}{D}$ represent the simplest form of the fraction and let $T = D - N$. For a digit A, if $.\overline{A}_T = 1 - .\overline{A}_T$ for numbers written in base T, compute A.

7. Let T = TNYWR and let $K = T + 3$. Compute the largest prime factor of $K^{2004} + 3K^{2002} + 2K^{2000}$.

8. Let m and n denote, respectively, the smaller and larger of the two numbers you will receive. Cubes A and B intersect each other and their intersection is cube C. The volume of C is m, the volume of A is at least as large as the volume of B, and the volume of the union of A and B is n. If the volumes of A and B are integers, compute the number of different volumes that cube A can have.

15. Compute the largest two-digit multiple of 3 that is one less than a perfect square.

14. Let T = TNYWR and let $K = T + 1$. The diagram shows ten congruent squares, each with an area of K. Compute the exact perimeter of $ABCDE$.



13. The number you will receive should be in the form $a + b\sqrt{2}$ where a and b are integers. If b consecutive integers have an arithmetic mean of $\frac{15}{2}$, compute the smallest of those integers.

12. Let T = TNYWR. The following system of equations has no solutions except for one particular value of K. Compute that value.

$$\begin{aligned} x + Ty + 5z &= -1 \\ -2x + 14y &= K + 10z \end{aligned}$$

11. Let T = TNYWR. *Andy's Ristorante of Miscellaneous Lunches* offers a special where you can order an entree, a dessert, and either a salad or an appetizer, but not both. There are 4 entrees, 3 desserts, 5 appetizers, 2 salads, and each salad will come with one of T different salad dressings. A customer must order an entree, but need not order any of the other dishes. Compute the number of different lunches that could be served.

10. Let T = TNYWR. Let $K = \frac{T}{5}$. Given $0° < \theta < 90°$, compute θ such that $\sin\left(\frac{\theta}{2} + K°\right) = \cos\frac{\theta}{6}$.

(Pass back the value of θ without the degree symbol).

9. Let T = TNYWR. Since $[\log T] = 1$, compute the value of $6 + [\log T^3]$. $[x]$ is the greatest integer function.

8. Let m and n denote, respectively, the smaller and larger of the two numbers you will receive. Cubes A and B intersect each other and their intersection is cube C. The volume of C is m, the volume of A is at least as large as the volume of B, and the volume of the union of A and B is n. If the volumes of A and B are integers, compute the number of different volumes that cube A can have.

1. 7
2. $12\sqrt{2}$
3. 16
4. 20.02
5. $\frac{10}{19}$
6. 4
7. 17

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15. 99

14. $100 + 30\sqrt{2}$

13. -7

12. 2

11. 160

10. 87°

9. 11

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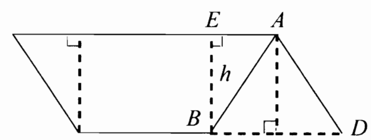
8. 4

1. The two-digit factors of 2002 are 11, 13, $2 \cdot 7 = 14$, $2 \cdot 11 = 22$, $2 \cdot 13 = 26$, $7 \cdot 11 = 77$, and $7 \cdot 13 = 91$. Ans: $\boxed{7}$.

2. Let r be the radius of the small circles → the radius of the large circle $= 3r$. Let M be the point of tangency of circle Q and $\overline{AB}$. Then $\overline{QM} \perp \overline{AB}$ and from right triangle MQB we have $r^2 + \left(\frac{AB}{2}\right)^2 = (3r)^2$. Since $T = 7, K = 9 = 3r$, then $r = 3 \rightarrow 9 + \left(\frac{AB}{2}\right)^2 = 81 \rightarrow AB^2 = 288 \rightarrow AB = \boxed{12\sqrt{2}}$.

3. $\sqrt{a^2 + b^2} = T$ and $\sqrt{a^2 + 4b^2} = \sqrt{T^2 + 96} \rightarrow a^2 + b^2 = T^2$ and $a^2 + 4b^2 = T^2 + 96$. Subtracting gives $3b^2 = 96 \rightarrow b^2 = 32 \rightarrow a^2 = T^2 - 32$. Since $T = 12\sqrt{2}$, then $a^2 = 288 - 32 = 256$.
Thus, $|a| = \boxed{16}$.

4. $ABCD$ is a rhombus whose area is $(1/2)(AC)(BD)$. Since $AE = \frac{T}{2}$, then $BD = T$. Let h be the height of the trapezoid. Then $(20.02)T = \frac{1}{2}(2h)(T) \rightarrow h = \boxed{20.02}$. T is irrelevant.



5. After the first marble is selected there are $K - 1$ marbles remaining and $\frac{K}{2}$ of them have a different color from the selected marble. So the probability is $\frac{K/2}{K-1}$. Since $T = 20.02$, then $K = 20$. Ans: $\boxed{\frac{10}{19}}$.

6. $1 - .\overline{A}_T = .\overline{A}_T \rightarrow .\overline{A}_T = \frac{1}{2}$. Thus, $\frac{A}{T} + \frac{A}{T^2} + \frac{A}{T^3} + \cdots = A\left(\frac{1/T}{1 - 1/T}\right) = \frac{A}{T-1} = \frac{1}{2}$. Possible answers (A, T) are (1, 3), (2, 5), (3, 7), (4, 9), . . . , (9, 19). Since $T = 9$, then $A = \boxed{4}$.

7. $K^{2004} + 3K^{2002} + 2K^{2000} = K^{2000}\left(K^4 + 3K^2 + 2\right) = K^{2000}\left(K^2 + 1\right)\left(K^2 + 2\right)$. Since $T = 4, K = 7$, and we have $7^{2000}(50)(51) = 7^{2000}(2 \cdot 5^2 \cdot 3 \cdot 17)$. Ans: $\boxed{17}$.

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15. Since $99 = 10^2 - 1$ and 99 is the largest two-digit multiple of 3, the answer is $\boxed{99}$.

14. Let x be the side of a square. Then the perimeter is $10x + 3x\sqrt{2}$. Since $T = 99$, $K = 100$, and $x^2 = 100 \rightarrow x = 10$. Answer: $\boxed{100 + 30\sqrt{2}}$.

13. Let the integers be $n, n+1, n+2, \ldots, n+(b-1) \rightarrow$ their sum $= bn + \frac{b(b-1)}{2}$. Dividing by b to obtain the mean yields $n + \frac{b-1}{2} = \frac{15}{2} \rightarrow n = \frac{16-b}{2}$. Since $T = 100 + 30\sqrt{2}$, $b = 30$ making $n = \boxed{-7}$.

12. The second equation can become $x - 7y + 5z = -\frac{K}{2}$. The equations are equivalent for $T = -7$ and $K = \boxed{2}$. The answer is independent of problem 13.

11. If no dessert is a choice, there are $3 + 1 = 4$ choices for dessert and $5 + 2T + 1$ choices for salad/appetizer or neither. The total number of meals is $4(5 + 2T + 1)4 = 96 + 32T$. Since $T = 2$, the answer is $\boxed{160}$.

10. Since $\cos\frac{\theta}{6} = \sin\left(90° - \frac{\theta}{6}\right) > 0$, we have $\sin\left(\frac{\theta}{2} + K°\right) = \sin\left(90° - \frac{\theta}{6}\right) \rightarrow \frac{\theta}{2} + K° = 90° - \frac{\theta}{6}$ or $\frac{\theta}{2} + K° = 180° - \left(90° - \frac{\theta}{6}\right)$. Thus, $\theta = \frac{3}{2}(90° - K°)$ or $3(90° - K°)$. Since $T = 160$, $K = 32$, making $\theta = 87°$ or $174°$. Since θ is acute, $\theta = \boxed{87°}$.

9. Since $[\log T] = 1$, T is a two digit number and $[\log T^3] = 3, 4$, or 5 for T in $[10, 21]$, $[22, 46]$, and $[47, 99]$ respectively. Since $T = 87$, $[\log T^3] = 5$ and $6 + [\log T^3] = \boxed{11}$.

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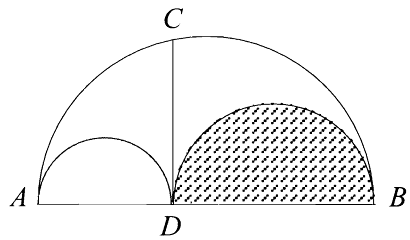
8. If A and B partially overlap, then $\text{vol}(A) + \text{vol}(B) - m = n \rightarrow \text{vol}(A) + \text{vol}(B) = n + m$. Since $\text{vol}(A) \geq \text{vol}(B)$, then $\text{vol}(A) \geq \frac{n+m}{2}$. If A and B coincide, then $\text{vol}(A) = \text{vol}(B) = n$ but since $n > m$, $\text{vol}(A) > \text{vol}(C)$. Thus, $m < \frac{n+m}{2} \leq \text{vol}(A) \leq n$. Since $m = 11$ and $n = 17$, we have $11 < 14 \leq \text{vol}(A) \leq 17$. So, the possible values for $\text{vol}(A)$ are 14, 15, 16, and 17. Answer: $\boxed{4}$.

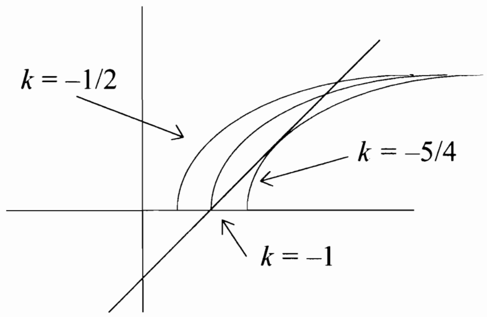
1. Compute all k such that the equation $\sqrt{x + k} = x - 1$ has two distinct real solutions for x.

2. Shown are three semicircles whose diameters lie on $\overline{AB}$. If $\overline{CD} \perp \overline{AB}$, $CD = 3$, and $AB = 10$, compute the area of the unshaded region.



3. There are n triangles of positive area that have one vertex at $A(0, 0)$ and the other two vertices at points with coordinates in $\{0, 1, 2\}$. Compute n.

1. $x + k = x^2 - 2x + 1 \rightarrow x^2 - 3x + (1 - k) = 0$. Thus, $9 - 4(1 - k) > 0 \rightarrow 4k > -5 \rightarrow k > \frac{-5}{4}$. But some of those solutions are extraneous once the curve $y = \sqrt{x + k}$ slides too far to the left as shown below:



Thus, $\boxed{-\frac{5}{4} < k \le -1}$.

2. Since ABC is a right triangle, $AD \cdot DB = CD^2$. Let $AD = x$, then $x(10 - x) = 9 \rightarrow x = 1$.

Then the unshaded area equals $\frac{1}{2}\pi \cdot 5^2 - \frac{1}{2}\pi \cdot \left(\frac{1}{2}\right)^2 - \frac{1}{2}\pi \cdot \left(\frac{9}{2}\right)^2 = \boxed{\frac{9\pi}{4}}$.

3. There are $3 \cdot 3 = 9$ points that can be formed using the coordinates $\{0, 1, 2\}$. Eliminate $A(0, 0)$ and from the remaining 8 points choose 2. This can be done in ${}_8C_2 = 28$ ways. Eliminate the pairs (0, 1) and (0, 2), (1, 1) and (2, 2), (1, 0) and (2, 0) since they are collinear with A, leaving $28 - 3 = \boxed{25}$ pairs of points.

ARML

2003

THE 28th ANNUAL MEET

This year there were 25 teams in Division A and 72 teams in Division B for a total of 99 teams involving close to 1600 students. There were teams from India and the Philippines but no teams from Taiwan this year due to SARS. There was a very spirited competition in Division A for the national championship. Going into the relays Thomas Jefferson trailed San Francisco Bay A by 6 points, 131 to 137, but with 24 points on the relays, TJ surged ahead, winning 155 to 153. In Division B Connecticut A won easily with 112 points, but there was spirited competition for second and third with 3 teams bunched very close together. Lehigh Valley prevailed for second place.

Sam Baethge of Texas received the received the Samuel Greitzer Distinguished Coach Award. Sam has long been associated with ARML and in 1997 he received the Alfred Kalfus Founder's Award in recognition of his stalwart and enthusiastic support.

Tatiana Shubin and Marilyn Blockus received the Alfred Kalfus Founder's Award for their fine and tireless work in setting up the new western site at San Jose State University.

The following received the Zachary Sobol Award for their outstanding contributions to their teams:

Dominic Albino	Western Massachusetts
Eve Drucker	AAST
Robert Ikeda	Southern California
David Vincent	Phillips Exeter

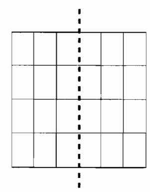
T–1. If $121_b = N_{10}$ and N has 9 factors, determine the least positive integer value for b.

T–2. Starting at 11:44 A.M., Tony walked for 5 miles. He noticed that his average speed, expressed in minutes and seconds per mile, was numerically equal to the time at which he stopped if he thought of the hour as minutes and the minutes as seconds. Compute the time at which he stopped.

T–3. $ABCD$ is a quadrilateral with $m\angle DAB = 90°$, $m\angle BCD = 135°$, $BC = 3$, and $CD = 2\sqrt{2}$. Compute the maximum possible area of $ABCD$.

T–4. Let $S = \{1, 2, 3, \ldots, 24, 25\}$. Compute the number of elements in the largest subset of S such that no two elements in the subset differ by the square of an integer.

T–5. A square is divided into 24 congruent rectangles as shown. On each side of the dotted line 4 rectangles are chosen at random and colored black. The square is then folded over the dotted line. Compute the probability that exactly one pair of black rectangles is coincident.



T–6. Let the faces of a unit cube be the six planes $x = 0, y = 0, z = 0, x = 1, y = 1, z = 1$. Compute the values of t such that the points $A(1, 1/2, t)$ and $B(1/2, 1, t)$ have <u>multiple</u> equal-length shortest paths connecting them along the faces of the unit cube.

T–7. In trapezoid $ABCD$, the perimeter is 600, all sides are integers, $AB = BC = CD$, and $\overline{AD}$ is the longest side. If the area of $ABCD$ equals $k\sqrt{k}$ for k an integer, compute k.

T–8. Let $2^x 3^y = (24^{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{60}})^1 \cdot (24^{\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{60}})^2 \cdot (24^{\frac{1}{4}+\frac{1}{5}+\cdots\frac{1}{60}})^3 \cdot \ldots \cdot (24^{\frac{1}{60}})^{59}$. Compute the value of $x + y$.

T–9. Assume that as a cubical bar of soap is used, all edges shrink at a constant rate of n units per day. Starting with a full bar, the soap was used for 6 days and its surface area was cut in half. Starting with a full bar of soap, compute exactly the time it would take for the volume to become one-eighth of the original volume.

T–10. ARMLovian, the language of the fair nation of ARMLovia, consists only of words using the letters A, R, M, and L. The words can be broken up into syllables that consist of exactly one vowel, possibly surrounded by a single consonant on either or both sides. For example, LAMAR, AA, RA, MAMMAL, MAMA, AMAL, LALA, MARLA, RALLAR, and AAALAAAAAMA are ARMLovian words, but MRLMRLM, MAMMMAL, MMMMM, L, ARM, ALARM, LLAMA, and MALL are not.

Compute the number of 7-letter ARMLovian words.

1. 5

2. 12:48

3. $\frac{41}{4} = 10.25$

4. 10

5. $\frac{224}{495} = .45\overline{25}$

6. $\frac{\sqrt{2}-1}{2}, \frac{3-\sqrt{2}}{2}$

7. 573

8. 3540

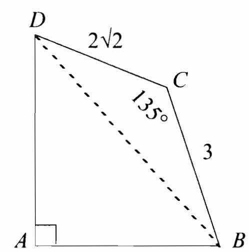
9. $6 + 3\sqrt{2}$

10. 1435

T–1. $121_b = b^2 + 2b + 1 = (b+1)^2$. Since $6^2 = 2^2 \cdot 3^2$ has $(2+1)(2+1) = 9$ factors and it is less than $2^8 = (2^4)^2$ which also has 9 factors, then $b = \boxed{5}$.

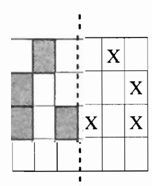
T–2. If he finished at x minutes past 12 then he walked for $16 + x$ minutes. Setting his average in minutes per mile equal to the time he stopped expressed as minutes gives $\frac{16+x}{5} = 12 + \frac{x}{60} \rightarrow 11x = 528 \rightarrow x = 48$. He stopped at $\boxed{12{:}48}$.

T–3. Break $ABCD$ into triangles ABD and BCD. The area of BCD is fixed and equals $\frac{1}{2} \cdot 3 \cdot 2\sqrt{2} \sin 135° = 3$. By the Law of Cosines, $DB^2 = 3^2 + \left(2\sqrt{2}\right)^2 - 2 \cdot 3 \cdot 2\sqrt{2} \cos 135° = 29 \rightarrow DB = \sqrt{29}$. Since DB is fixed, the area of ABD is maximized when ABD is an isosceles right triangle, making $AD = AB = \frac{\sqrt{29}}{\sqrt{2}} \rightarrow$ the maximum area of $ABD = \frac{29}{4}$. Thus, the maximum area of $ABCD$ equals $3 + \frac{29}{4} = \boxed{\frac{41}{4}}$.



T–4. Note that of any five consecutive numbers $k, k+1, k+2, k+3, k+4$ at most two of them can be in S. If three were in S, either two of them would be adjacent and would differ by 1, a square, or they would be $k, k+2, k+4$. But $k+4$ and k differ by a perfect square. Apply this lemma to the subsets $\{1, \ldots, 5\}$, $\{6, \ldots, 10\}$, $\{11, \ldots, 15\}$, $\{16, \ldots, 20\}$, and $\{21, \ldots, 25\}$. At most two of each set can be in S, so at most 10 of the elements of $\{1, 2, \ldots, 25\}$ can be in S. The set $\{1, 3, 6, 8, 11, 13, 16, 18, 21, 23\}$ clearly satisfies the conditions, so the answer is $\boxed{10}$.

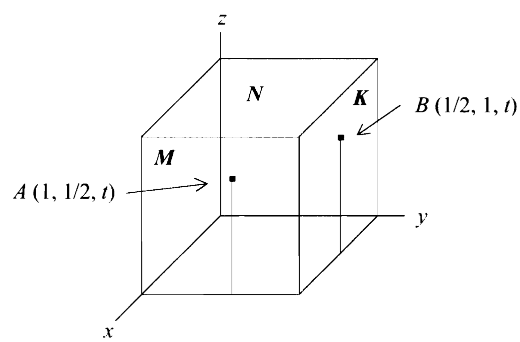
T–5. There are $\binom{12}{4}$ ways to choose 4 rectangles on the left. For exactly one rectangle to be coincident with one on the other side, we must choose one of the four possible X's and that can be done in $\binom{4}{1}$ ways. We must choose 3 blank rectangles and that can be done in $\binom{8}{3}$ ways. Answer: $\frac{\binom{4}{1} \cdot \binom{8}{3}}{\binom{12}{4}} = \boxed{\frac{224}{495}}$.

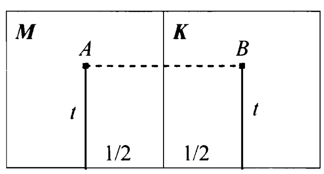


Alternate solution: There are $\binom{12}{4}^2$ ways to choose 4 rectangles on each side of the dotted line. Letting lower case letters stand for the rectangles, let $A = \{a, x, y, z\}$ be the rectangles chosen on one side and $B = \{a, u, v, w\}$ be the rectangles chosen on the other. We need x, y, z, u, v, w to be distinct. There are 12 choices for a, $\binom{11}{3}$ ways to choose x, y, and z, and $\binom{8}{3}$ ways to choose u, v, and w. Thus,

$$\frac{12\cdot\binom{11}{3}\cdot\binom{8}{3}}{\binom{12}{4}^2} = \frac{224}{495}.$$

T–6. Consider the cube in the diagram to the right and consider the case where $t > \frac{1}{2}$. Let M, N, and K be the names of the faces. Unfold the cube in the two ways shown below.





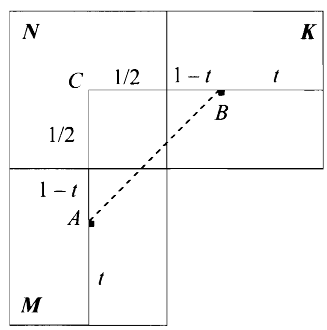


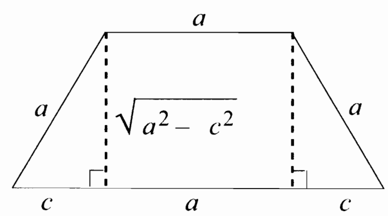
Fig. 1 Fig. 2

In Fig. 1 the distance between A and B is clearly 1. So, in Fig. 2 the distance must be 1 as well, but it also equals $\sqrt{2}\left(\frac{3}{2} - t\right)$ since ABC is a 45-45-90 right triangle and $AC = \frac{1}{2} + (1 - t)$. Thus,

$\sqrt{2}\left(\frac{3}{2} - t\right) = 1 \rightarrow t = \frac{3-\sqrt{2}}{2}$. If $t < \frac{1}{2}$, then $1 - t$ would give the same two equal length paths.

Thus, $1-\left(\frac{3-\sqrt{2}}{2}\right) = \frac{\sqrt{2}-1}{2}$ also works. If $t = \frac{1}{2}$, the distance from A to B in Fig. 1 equals 1, while in Fig. 2 it equals $\sqrt{2}$. Thus, A and B can't lie in the center of a face. Answer: $t = \boxed{\frac{\sqrt{2}-1}{2} \text{ or } \frac{3-\sqrt{2}}{2}}$.

T–7. $P = 600 = 4a + 2c \rightarrow 300 = 2a + c$. The area equals $\frac{1}{2}(2a+2c)\sqrt{a^2-c^2} = (a+c)\sqrt{a^2-c^2} = k\sqrt{k} \rightarrow$ $(a-c)(a+c)^3 = k^3$. Since a, c, and k are positive integers, $a - c = m^3$ for some positive integer m.



From $300 = 2a + c$ and $a - c = m^3$, we obtain $3a = 300 + m^3$, giving $a = \frac{m^3}{3} + 100$. Let $m = 3n$ $\rightarrow a = 9n^3 + 100$ and $c = 100 - 18n^3$. For $n \geq 2$, $c < 0$, so for $n = 1$, $a = 109$, $c = 82$, giving $k = \boxed{573}$.

T–8. Observe that

$$\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{60}\right)+\left(\frac{2}{3}+\frac{2}{4}+\cdots+\frac{2}{60}\right)+\left(\frac{3}{4}+\frac{3}{5}+\cdots+\frac{3}{60}\right)+\cdots+\left(\frac{58}{59}+\frac{58}{60}\right)+\frac{59}{60}$$

$$= \frac{1}{2}+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\cdots+\left(\frac{1}{60}+\frac{2}{60}+\cdots+\frac{59}{60}\right)$$

$$= \frac{1}{2}+\left(\frac{2\cdot 3}{2\cdot 3}\right)+\left(\frac{3\cdot 4}{2\cdot 4}\right)+\left(\frac{4\cdot 5}{2\cdot 5}\right)+\cdots+\left(\frac{59\cdot 60}{2\cdot 60}\right) = \frac{1+2+3+\cdots+59}{2} = 885.$$

Thus, $2^x3^y = 24^{885} = \left(2^3 3\right)^{885} \rightarrow x = 3(885)$ and $y = 885$. Thus, $x + y = \boxed{3540}$.

T–9. Since in 6 days the surface area $6a^2$ becomes $3a^2 = 6\left(\frac{a}{\sqrt{2}} \cdot \frac{a}{\sqrt{2}}\right)$, then in 6 days each side must be $1/\sqrt{2}$ as long. To reach a volume of one-eighth of the original, each side must be one-half as long. Without loss of generality, let the bar be a cube of side 1. Consider the linear function where edge y is a function of time t. At $t = 0$, $y = 1$ and at $t = 6$, $y = 1/\sqrt{2}$. The slope of this line is $\frac{1-\sqrt{2}}{6\sqrt{2}}$ and its equation is $y - 1 = \left(\frac{1-\sqrt{2}}{6\sqrt{2}}\right)t$. Let $y = 1/2$ and solve for t obtaining $\boxed{t = 6 + 3\sqrt{2}}$ days. This is about 10.24 days.

T–10. The conditions given are equivalent to the condition that a word not start or end with more than one consonant and not have a consecutive string of three or more consonants. We'll count the number of possible n-letter words ending in a vowel V_n, one consonant O_n, or two consonants T_n. We include the words ending in two consonants because although they are not ARMLovian words, they generate ARMLovian words by the addition of a vowel at the end.

Note that $V_1 = 1, O_1 = 0$. Also, $V_2 = 4$, i.e., AA, LA, MA, and RA, $O_2 = 3$, i.e., AR, AM, and AL, and $T_2 = 0$ since no word can end in two consonants. In general, $V_n = V_{n-1} + O_{n-1} + T_{n-1}$ because adding a vowel to the end of a word ending in a vowel, 1 consonant, or 2 consonants will produce an ARMLovian word. Or one could say that removing a vowel from the end of an ARMLovian word will yield a prefix ending in 0, 1 or 2 consonants.

In addition, $O_n = 3V_{n-1}$ since adding any one of three consonants to the end of a word ending in a vowel will produce an ARMLovian word. Finally, $T_n = 3 \cdot O_{n-1}$ since any one of three consonants can be added to the end of a word ending in a consonant. Thus, we can generate the following values:
$V_3 = V_2 + O_2 + T_2 = 4 + 3 + 0 = 7$, $O_3 = 3V_2 = 3 \cdot 4 = 12$, and $T_3 = 3 \cdot O_3 = 9$.
Continuing in this fashion we obtain:

$V_4 = 28$	$O_4 = 21$	$T_4 = 36$
$V_5 = 85$	$O_5 = 84$	$T_5 = 63$
$V_6 = 232$	$O_6 = 255$	$T_6 = 252$
$V_7 = 739$	$O_7 = 696$	$T_7 = 765$

The number of acceptable seven-letter words is $V_7 + O_7 = 739 + 696 = \boxed{1435}$.

Alternate solution: Let's determine the number of words by the number of A's in the word:

7 A's 1 word, namely AAAAAAA.

6 A's 21 words since there are 3 choices for the consonant and it can go in one of 7 places. Thus, $3\left({}_7C_1\right) = 3 \cdot 7 = 21$.

5 A's 171 words. There are 3 · 3 choices for a consonant and the number of choices for their position is ${}_7C_2 - 2$ since they can go anywhere except together in the first two positions or together in the last two positions. Thus, $9 \cdot \left({}_7C_2 - 2\right) = 9 \cdot 19 = 171$.

4 A's 594 words. There are 3^3 choices for the three consonants. They can't be together in the first two positions, together in the last two positions, or be three in a row. If the consonants were 3 in a row, they could occupy positions 1 – 3, 2 – 4, . . . , 5 – 7 giving 5 possible placements. If they occupied the first two positions but not the 3rd, there would be 4 possible placements of the consonants. Similarly if they occupied the last two positions. Thus, there are 5 + 4 + 4 = 13 positions the consonants cannot occupy. This gives $27 \cdot \left({}_7C_3 - 13\right) = 27 \cdot 22 = 594$ ARMLovian words with 4 vowels.

3 A's 648 words. There are 3^4 ways to choose the consonants. Once chosen we have the following possible words:

AccAccA, AcAccAc, AccAcAc, cAccAcA,
cAcAccA, cAAccAc, cAcAcAc, and cAccAAc.

This gives 8 · 81 = 648.

There are no ARMLovian words with 2 or fewer A's since that would put 2 consonants together at the beginning or end, or would give a run of 3.

The total number of words is 648 + 594 + 171 + 21 + 1 = 1435.

A set of positive integers $\{x_1, x_2, \ldots\}$ is called a *Fibonacci set* if $x_1 < x_2$ and $x_n = x_{n-1} + x_{n-2}$ for all $n > 2$.

We say that $\{a, b\}$ is a subset of a Fibonacci set *F* if *a* and *b* are distinct but not necessarily consecutive members of *F*. Notice that any two-member set $\{a, b\}$ of positive integers is a subset of at least one Fibonacci set. Since any Fibonacci set is determined by specifying its two smallest members, list just the first two members. Thus, write $\{1, 2, \ldots\}$ for $\{1, 2, 3, 5, 8, 13, \ldots\}$.

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1. a) Give an example of a set of positive integers $\{a, b\}$ that is a subset of <u>only one</u> Fibonacci set.

   b) For how many Fibonacci sets $\{x_1, x_2, \ldots\}$ is it true that $x_3 = 2003$?

2. a) Find all Fibonacci sets that have $\{8, 144\}$ as a subset. Specify each one by listing its two smallest members.

   b) Let *a* and *b* be positive integers with $0 < a < b$. Prove that $\{a, b\}$ is a subset of more than one Fibonacci set if and only if $2a < b$ or $2a > b$.

3. Prove that no Fibonacci set has $\{60, 117, 174\}$ as a subset.

4. Compute the number of Fibonacci sets where $x_n = 2003$ for $n \geq 2$.

5. Prove that any Fibonacci set has infinitely many subsets of the form $\{a, a + d, a + 2d\}$.

6. Prove that no Fibonacci set has a subset of the form $\{a, a + d, a + 2d, a + 3d\}$.

A set of positive integers $\{x_1, x_2, \ldots\}$ is called a *linear set* if there is a positive integer d for which $x_n = x_{n-1} + d$ for all $n > 1$.

7. a) Prove that the Fibonacci set {7, 11, . . . } and the linear set {8, 23, . . . } are disjoint.

   b) Is it possible for a Fibonacci set and a linear set to have members in common, but only *finitely many*? Prove your answer.

8. a) Prove that two Fibonacci sets can have exactly one member in common.

   b) Prove that two Fibonacci sets can have exactly two members in common.

9. a) Prove that whenever two Fibonacci sets have more than two members in common, then the set of common members is itself a Fibonacci set.

   b) Prove: Given any Fibonacci sets $F_1, F_2, \ldots, F_n$, there is a positive integer that does not belong to any of them.

10. Can the set of positive integers be partitioned into Fibonacci sets? This means finding infinitely many pair-wise disjoint Fibonacci sets whose union contains every positive integer. Prove your answer.

1. a) Without loss of generality assume that $a < b$. The simplest example is, of course, $\{1, 2, \ldots\}$.

   b) Let x_1 be any integer between 1 and 1001 inclusive and let $x_2 = 2003 - x_1$. For all of these sets $\{x_1, 2003 - x_1, \ldots\}$ the third element will be 2003. There are 1001 such sets, ranging from $\{1, 2002, \ldots\}$ to $\{1001, 1002, \ldots\}$.

2. a) There are 9 such Fibonacci sets: $\{1, 2, \ldots\}$, $\{2, 3, \ldots\}$, $\{3, 5, \ldots\}$, $\{5, 8, \ldots\}$, $\{8, 13, \ldots\}$, $\{8, 24, \ldots\}$, $\{8, 68, \ldots\}$, $\{8, 136, \ldots\}$, and $\{8, 144, \ldots\}$.

   Justification: Let x be the term that immediately follows 8 in any Fibonacci set F that contains 8. Then the members of F following 8 must be x, $8 + x$, $8 + 2x$, $16 + 3x$, $24 + 5x$, $40 + 8x$, $64 + 13x, \ldots$. Since $x > 8$, $64 + 13x > 168$ and we need only consider the first 6 expressions. Setting each in turn equal to 144 we have $x = 144$, $8 + x = 144$, $8 + 2x = 144$, $16 + 3x = 144$, $24 + 5x = 144$, and $40 + 8x = 144$. These yield respectively $x = 144, 136, 68, 42 + 2/3, 24$, and 13. Thus, $16 + 3x$ can't serve as a term, giving at least 5 Fibonacci sets containing both 8 and 144. These are $\{8, 144, \ldots\}$, $\{8, 136, \ldots\}$, $\{8, 68, \ldots\}$, $\{8, 24, \ldots\}$, and $\{8, 13, \ldots\}$. In the first four sets, 8 is less than half of x, so 8 and x are the smallest members of F. In the fifth set where $x = 13$, 8 can have predecessors and F could begin with two consecutive terms chosen from the standard Fibonacci set $\{1, 2, 3, 5, 8, 13, \ldots\}$.

   b) First, if $b \neq 2a$, then either $b < 2a \rightarrow b - a < a$, in which case let $F = \{b - a, a, b, \ldots\}$, or $2a < b \rightarrow a < b - a$, in which case let $F = \{a, b - a, b, \ldots\}$. In other words, $b < 2a$ means that there is room for a to have a predecessor and $2a < b$ means that there is room for a term between a and b. In neither case are a and b the two smallest numbers. Conversely, we will show that if a and b are not the two smallest numbers, then $b \neq 2a$. If F contains a member m between a and b, then $2a < a + m \leq b \rightarrow 2a < b$. If F does not have members between a and b, but does have a member x smaller than a, then $b = x + a < 2a \rightarrow b < 2a$. So, if a and b are not the two smallest numbers, then $b \neq 2a$.

3. Referring to the proof in (2b) we conclude that because 117 is more than half of 174, no Fibonacci set that contains 117 and 174 can have members between 117 and 174. Thus, the member that immediately precedes 117 must be $174 - 117 = 57 < 60$.

4. If $x_2 = 2003$, then $x_1 = 1, 2, \ldots, 2002$ all work, giving 2002 sets.

If $x_3 = 2003$, then we know from (1b) that there are 1001 sets.

If $x_4 = 2003 = 2x_2 + x_1$, we obtain {1, 1001, . . .}, {3, 1000, . . .}, . . . {667, 668, . . .} for a total of 334 sets.

If $x_5 = 2003 = 3x_2 + 2x_1$, we obtain {1, 667, . . .}, {4, 665, . . .}, . . . , {400, 401, . . .} for a total of 134 sets.

If $x_6 = 2003 = 5x_2 + 3x_1$, we obtain {1, 400, . . .}, {6, 397, . . .}, . . . , {246, 253, . . .} for a total of 50 sets.

If $x_7 = 2003 = 8x_2 + 5x_1$, we obtain {7, 246, . . .}, {15, 241, . . .}, . . . , {151, 156, . . .} for a total of 19 sets.

If $x_8 = 2003 = 13x_2 + 8x_1$, we obtain {5, 151, . . .}, {18, 143, . . .}, . . . , {83, 103, . . .} for a total of 7 sets.

If $x_9 = 2003 = 21x_2 + 13x_1$, we obtain {20, 83, . . .} and {41, 70, . . .} for a total of 2 sets.

If $x_{10} = 2003 = 34x_2 + 21x_1$, we obtain only {29, 41, . . .}.

If $x_{11} = 2003 = 55x_2 + 34x_1$, we obtain only {12, 29, . . .}.

<u>Justification</u>: Each equation for x_i in terms of x_1 and x_2 is a linear Diophantine equation, solvable by writing x_1 and x_2 in terms of a parameter t. For example, from $2003 = 8x_2 + 5x_1$ we obtain

$x_1 = 400 + \frac{3}{5} - x_2 - \frac{3x_2}{5} = 400 - x_2 + \frac{3 - 3x_2}{5}$. Let $3 - 3x_2 = 5m$ for m an integer, giving

$x_1 = 400 - x_2 + m$ and $x_2 = 1 - m - \frac{2m}{3}$. Let $m = 3t$ for t an integer and then $x_2 = 1 - 3t - 2t = 1 - 5t$ and $x_1 = 400 - (1 - 5t) + 3t = 399 + 8t$. Thus, if $x_1 = 399 + 8t$ and $x_2 = 1 - 5t$ for t an integer, we find solutions by choosing all values of t for which $0 < x_1 < x_2$, namely for $0 < 399 + 8t < 1 - 5t$ which gives $-49.875 < t < -30.615 \rightarrow -49 \le t \le -31$. Since there are 19 values of t generating proper values (x_1, x_2), there are 19 solutions. Experience with linear Diophantine equations teaches that the x_1 values increase by the coefficient of x_2 and the x_2 values decrease by the coefficient of x_1 and this would provide a quicker way to generate solutions once one has been found. For example, once we discover that the first solution to $2003 = 8x_2 + 5x_1$ is {7, 246} then we could write $x_1 = 7 + 8k$ and $x_2 = 246 - 5k$, determine the appropriate values of k and find the number of solutions.

Interestingly, the x_1 in the last ordered pair for x_n is the x_2 value for the first ordered pair for x_{n+1} and the x_1 value for the first term for x_{n+1} equals $x_2 - x_1$ using the values for the last term in x_n. That would imply that $x_{12} = \{17, 12\}$, a contradiction since that makes $x_1 > x_2$.

Adding, we find there are 3551 Fibonacci sets with $x_n = 2003$ for $n \geq 2$.

5. Given any four consecutive members of a Fibonacci set, the third is the average of the 1st and 4th. In other words, given $a, x, a+x, a+2x$, we have $a+x = \dfrac{a+(a+2x)}{2}$. This says that x_i, x_{i+2}, x_{i+3} form an arithmetic sequence for any Fibonacci set for any $i \geq 1$.

6. Suppose there is a four term arithmetic subsequence of a Fibonacci set. Call it $\{a, a+x, a+2x, a+3x\}$. By (2b) if twice the smaller is greater than the larger, then there are no intervening terms. Since $2(a+2x) > a+3x$ and $2(a+x) > a+2x$, then the elements $a+x, a+2x$, and $a+3x$ have no intervening terms. Thus, they are consecutive elements of a Fibonacci set, making $(a+x)+(a+2x) = a+3x$, and hence $a = 0$. But this is a contradiction, showing that a four term arithmetic sequence is impossible in a Fibonacci set.

7. a) Modulo 15, the members of the Fibonacci set are periodic and are congruent to 7, 11, 3, 14, 2, 1, 3, and 4. Modulo 15, the members of the linear set are all congruent to 8. Therefore, there are no members in common.

b) It is impossible for a Fibonacci set and a linear set to have a member in common without having infinitely many members in common. To see why, suppose that d is the positive difference between consecutive members of the linear set and that all members of the linear set are congruent to k modulo d where $k < d$. If the members of the Fibonacci set are calculated modulo d (listing just remainders), the list will be *periodic*. This is because there are only d different remainders, hence at most d^2 pairs of consecutive terms that can occur in such a list. This implies that some pair must appear for a second time. Suppose that the members are $x_1, x_2, x_3, \ldots$, and that x_i is congruent to x_n, and that x_{i+1} is congruent to x_{n+1} (modulo d) for some indices $i < n$. If $1 < i$, then x_{i-1} is also congruent to x_{n-1} because $x_{i-1} = x_{i+1} - x_i$ and $x_{n-1} = x_{n+1} - x_n$ (modulo d). It follows that the initial pair of x-values must, in fact, repeat, and so too must *every* x-value in the list. Thus, if the linear value k appears once in the list, then it appears infinitely often.

8. a) If $x_1 = y_1$ and $x_2 < y_2 < x_3$, then the Fibonacci sets $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ have only x_1 in common. In fact, $x_i < y_i < x_{i+1} < y_{i+1}$ holds for all indices $1 < i$. This can be shown by induction. Notice first that $x_1 = y_1$ and $x_2 < y_2$ imply that $x_3 < y_3$ and that initializes the induction. The inductive step is also a consequence of the Fibonacci recursion, namely $x_i < y_i < x_{i+1} < y_{i+1}$ implies $x_{i+2} < y_{i+2}$ and $y_{i-1} < x_i < y_i < x_{i+1}$ implies $y_{i+1} < x_{i+2}$. Combine the data to obtain $x_{i+1} < y_{i+1} < x_{i+2} < y_{i+2}$.

If Fibonacci sets $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ satisfy $x_i < y_1 < x_{i+1} < y_2 < x_{i+2}$ for some index *i*, then the preceding shows that the sets are *interleaved*—neither set has two members between consecutive members of the other set.

b) Given any Fibonacci set $\{x_1, x_2, \ldots\}$, let $y_1 = x_i$ and $y_2 = x_j$ for any positive index *i* and any index *j* that is greater than $i + 1$. The Fibonacci sets $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ have only y_1 and y_2 in common because they interleave $x_j < y_3 < x_{j+1} < y_4 < x_{j+2}$ after the two common members. The first inequality $x_j < y_3$ is obvious, the second follows from $y_3 = x_i + x_j < x_{j-1} + x_j = x_{j+1}$, the third follows from $x_{j-1} < y_2$ and $x_j < y_3$, and the fourth follows from $y_2 = x_j$ and $y_3 < x_{j+1}$.

9. a) Let *a* and *b* be the smallest two integers that the Fibonacci sets $F = \{x_1, x_2, \ldots\}$ and $G = \{y_1, y_2, \ldots\}$ have in common with $a < b$. If *a* and *b* are consecutive members of both *F* and *G*, then *F* and *G* share all their members beyond *b* and the desired conclusion follows easily. Suppose, therefore, that *a* and *b* are not consecutive members of *F* which implies that $a < x_{i-1} < x_i = b = y_j$. There are two cases: $x_{i-1} < y_{j-1}$ or $y_{j-1} < x_{i-1}$.

In the first case the Fibonacci recursion implies that $y_j < x_{i+1} < y_{j+1} < x_{i+2} < y_{j+2}$.

In the second case the recursion implies that $x_i < y_{j+1} < x_{i+1} < y_{j+2} < x_{i+2}$.

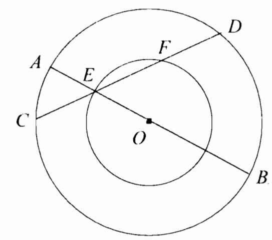
In either case, *F* and *G* have only *a* and *b* in common. It follows that *F* and *G* have more than two members in common only if the two smallest shared members are consecutive members of both sets, in which case the Fibonacci set generated by the two smallest members equals the set of shared members.

b) Here the following fact is needed: If $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ are Fibonacci sets and if there are more than two y-values between x_i and x_{i+1}, then $i = 1$. In other words, only the initial interval of a Fibonacci set can include more than two members of any other Fibonacci set. The reason is that the first two of those y-values would otherwise be greater than x_{i-1} and x_i respectively, hence the third y-value would have to be greater than x_{i+1} by the Fibonacci recursion.

For any Fibonacci set, the recursion $x_n - x_{n-1} = x_{n-2}$ shows that each member is equal to the difference between the next member and the one following. Thus this "gap" is an increasing function of position. No matter how large n is, the gaps in F_1 will eventually be larger than $2n$. And as shown previously, each gap can contain at most two members from each of the other Fibonacci sets, implying that there are integers in that gap that belong to none of the other F_i.

10. The set of positive integers can be partitioned into infinitely many Fibonacci sets. Let $F_1 = \{1, 2, 3, 5, \ldots\}$, $F_2 = \{4, 6, 10, 16, \ldots\}$, and $F_3 = \{7, 11, 18, 29, \ldots\}$. Notice that each F_i for $i = 1, 2$, or 3 is interleaved with the other two and therefore the three sets are mutually disjoint. Assume inductively that the Fibonacci sets $F_1, F_2, \ldots, F_n$ have been chosen by the following procedure: Let m be the smallest positive integer that is not a member of any of the F_i. Thus, every F_i has at least one member that is smaller than m. In particular, $m - 1$ is a member x_i of a Fibonacci set F_c. Let $p = 1 + x_{i+1}$. The desired F_{n+1} is defined by its first two members m and p and it is interleaved with every previously defined Fibonacci set because the interleaved sets $F_1, F_2, \ldots, F_n$ each have one member between x_i and x_{i+1}.

I–1. O is the center of two concentric circles as shown. $\overline{AB}$ and $\overline{CD}$ are chords of the larger circle and they intersect at E on the smaller circle. $\overline{CD}$ intersects the smaller circle at F. If $m\widehat{AC} + m\widehat{EF} = 83°$, compute $m\widehat{DB}$.



I–2. A triangle with sides $a \le b \le c$ is log–right if $\log\left(a^2\right) + \log\left(b^2\right) = \log\left(c^2\right)$. Compute the largest possible value of a in a right triangle that is also log–right.

I–3. Compute the least prime p such that $p - 1$ equals the difference of the squares of two positive multiples of 4.

I–4. Let N be a three-digit number that is divisible by 3. One of N's digits is chosen at random and removed. Compute the probability that the remaining number is divisible by 3. (Note: if a digit is removed from 207 we obtain either 20, 07 = 7, or 27; if a digit is removed from 300 we obtain either 30, 30, or 00 = 0.)

I–5. There are n triangles of positive area that have one vertex at $A(0, 0)$ and the other two vertices at points whose coordinates are drawn independently and with replacement from $\{0, 1, 2, 3, 4\}$. Compute n.

I–6. Let n be an integer. Of all fractions $\frac{1}{n}$, the fractional part of $\sqrt{123456789}$ is closest to one such fraction. Compute that value of n.

I–7. Compute the largest factor of 1001001001 that is less than 10,000.

I–8. The graph of $f(x) = x^4 + 4x^3 - 16x^2 + 6x - 5$ has a common tangent line at $x = p$ and $x = q$. Compute the product pq.

1. 97°

2. $\sqrt{2}$

3. 113

4. $\dfrac{19}{50} = .38$

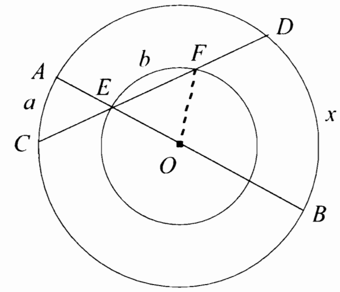
5. 256

6. 9

7. 9901

8. –10

I–1. Let $m\widehat{AC} = a$, $m\widehat{EF} = b$, and $m\widehat{DB} = x$. Then



$m\angle EOF = b \rightarrow m\angle FOB = 180 - b \rightarrow m\angle FEO = \dfrac{180 - b}{2}$. But

$m\angle FEO = \dfrac{a + x}{2} \rightarrow 180 - b = a + x \rightarrow x = 180 - (a + b)$.

Since $a + b = 83°$, $x = \boxed{97°}$.

I–2. From $\log a^2 + \log b^2 = \log c^2$ we obtain $ab = c$. Substituting into $a^2 + b^2 = c^2$ yields $(ab)^2 = a^2 + b^2$ $\rightarrow a^2b^2 - b^2 = a^2$. Since $a \le b$, replacing b by a yields the following inequality: $a^2a^2 - a^2 \le a^2$. Thus, $a^4 \le 2a^2 \rightarrow a^2 \le 2 \rightarrow a \le \sqrt{2}$. Thus, the largest value of a is $\boxed{\sqrt{2}}$. If $a = b = \sqrt{2}$, then $c = 2$ and both the Pythagorean Theorem and $\log a^2 + \log b^2 = \log c^2$ are satisfied.

I–3. $p - 1 = (4k)^2 - (4m)^2 = 16(k^2 - m^2)$. If k and m are odd, their difference is divisible by 8, making $p = 128t + 1$. The least prime is 257 obtained for $t = 2$, $k = 5$ and $m = 3$. If one is odd and the other even, then their difference is odd, making $p = 16(2t + 1) + 1$ and the least prime is 113 obtained when $t = 3$, $k = 4$ and $m = 3$. If both are even, then $p = 64t + 1$ and the least prime is 193 obtained when $t = 3$, $k = 4$ and $m = 2$. Thus, the least prime is $\boxed{113}$.

I–4. There are 300 three-digit numbers times 3 digits = 900 cases. If the numbers are written as 3xx, 6xx, or 9xx and one of 3, 6, or 9 is removed, then since there are 34 numbers divisible by 3 from 00 to 99, there are $3 \cdot 34 = 102$ possible favorable outcomes. If the numbers are of the form x0x, x3x, x6x, or x9x, then removing 0, 3, 6, or 9 yields a number divisible by 3 in the 30 cases from 10 to 99, giving $4 \cdot 30 = 120$ favorable outcomes. Similarly for xx0, xx3, xx6, and xx9. Thus, there are $102 + 2 \cdot 120 = 342$ favorable outcomes yielding a probability of $\dfrac{342}{900} = \boxed{\dfrac{19}{50}}$.

I–5. There are $5 \cdot 5 = 25$ points that can be formed using the coordinates $\{0, 1, 2, 3, 4\}$. Eliminate $A(0, 0)$ and from the remaining 24 points, choose 2. This can be done in ${}_{24}C_2 = (24 \cdot 23)/2 = 276$ ways. Pairs such as (0,1) and (0, 2), (1, 1) and (2, 2) are collinear with $A(0, 0)$ and must be eliminated. Thus, subtract $3 \cdot {}_4C_2 = 18$ points to eliminate those pairs of points on $x = 0, y = x$, and $y = 0$, but we must also eliminate the two pairs (1, 2), (2, 4) and (2, 1), (4, 2) for a total of $276 - 18 - 2 = \boxed{256}$ triangles.
Note: if the set was $\{0, 1, 2\}$ there are 16 triangles. Is there a pattern here?

I–6. Let $x = \sqrt{123456789}$. Then $\frac{x}{10^5} = \sqrt{0.0123456789} \approx \sqrt{\frac{1}{10^2} + \frac{2}{10^3} + \frac{3}{10^4} + \ldots}$. To evaluate this, set

$$S = \frac{1}{10^2} + \frac{2}{10^3} + \frac{3}{10^4} + \frac{4}{10^5} + \cdots,$$ then divide by 10 giving

$$\frac{S}{10} = \frac{1}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{4}{10^6} + \cdots$$

Thus, $$\frac{9}{10}S = \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots = \frac{\frac{1}{100}}{1 - \frac{1}{10}} = \frac{1}{90} \rightarrow S = \frac{1}{81}.$$ This gives

$\frac{x}{10^5} \approx \frac{1}{9} \rightarrow x \approx \frac{100000}{9} = 11111 + \frac{1}{9}$. Hence, $n = \boxed{9}$.

<u>Alternate solution:</u> One could observe that $11111^2 = 123454321$ and write

$$\frac{1}{\sqrt{123456789} - 11111} = \frac{\sqrt{123456789} + 11111}{123456789 - 11111^2} \approx \frac{22222}{2468} \approx 9.$$

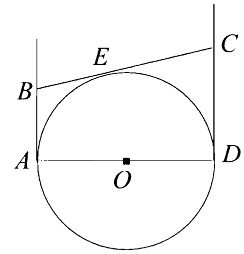
I–7. $1001001001 = 1001 \cdot 10^6 + 1001 = (1001)(10^6 + 1) = 1001(10^2 + 1)(10^4 - 10^2 + 1) = 1001 \cdot 101 \cdot 9901$ $= 7 \cdot 11 \cdot 13 \cdot 101 \cdot 9901$. Since no combination of 7, 11, 13, and 101 can generate a factor greater than 9901 but less than 10,000, the answer is $\boxed{9901}$.

I–8. Let the equation of the common tangent be $y = mx + b$. Consider the function $g(x) = x^4 + 4x^3 - 16x^2 + 6x - 5 - (mx + b)$. It must have double zeros at $x = p$ and q. Thus, $x^4 + 4x^3 - 16x^2 + (6 - m)x - (5 + b) = (x - p)^2(x - q)^2 =$ $x^4 - 2(p + q)x^3 + (p^2 + 4pq + q^2)x^2 - 2(pq^2 - p^2q)x + p^2q^2$. Setting the coefficients of x^3 equal gives $-2(p + q) = 4 \rightarrow p + q = -2$. Setting the coefficients of x^2 equal gives $p^2 + 4pq + q^2 = -16 \rightarrow$ $(p + q)^2 + 2pq = -16 \rightarrow 4 + 2pq = -16 \rightarrow pq = \boxed{-10}$. In this case, the common tangent is $y = 46x - 105$ and letting $p > q$, we have $p = -1 + \sqrt{11}$ and $q = -1 - \sqrt{11}$.

ARML Relay #1 – 2003

R1–1. Mary insists on sitting next to Bob and Bob insists on sitting next to Jane. Compute the number of different ways that Bob, Mary, Jane, and 3 other students can sit in a row.

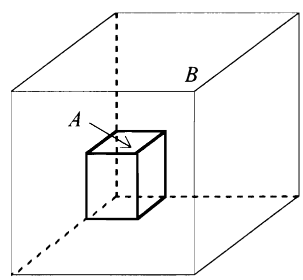
R1–2. Let T = TNYWR. $\overline{AD}$ is a diameter of circle O and has length $\frac{T}{3}$. $\overline{BA}$, $\overline{CD}$, and $\overline{BC}$ are tangent to circle O at A, D, and E, respectively. If $BC = 18$, compute the area of $ABCD$.



R1–3. Let T = TNYWR. The circle $x^2 + y^2 = T$ intersects the positive y- and x-axis at A and B respectively. If the line $y = 3$ intersects $\overline{AB}$ at C, compute the larger of AC or BC.

ARML Relay #2 – 2003

R2–1. A box 5 by 8 by 12 is placed in the corner of a cube of side 14 so that three faces of the box are coincident with three faces of the cube. Compute the distance from corner A to corner B.



R2–2. Let T = TNYWR and let $K = T - 7$. If $\log_{16} K$, $\log_K 16$, and x form a geometric progression, compute x.

R2–3. Let T = TNYWR. In ΔABC, $m\angle B = 90°$, and D lies on $\overline{CB}$ so that $\overline{AD}$ bisects $\angle CAB$ and $CD = T \cdot (DB)$. Compute $\cos^2 \angle CAD$.

Relay #1:

R1–1. 48

R1–2. 144

R1–3. $9\sqrt{2}$

Relay #2:

R2–1. 11

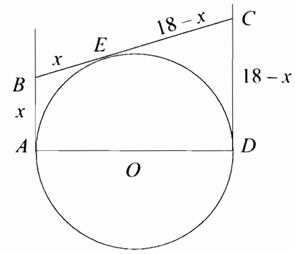
R2–2. 8

R2–3. $\frac{9}{16} = .5625$

Solutions to the ARML Relay #1 – 2003

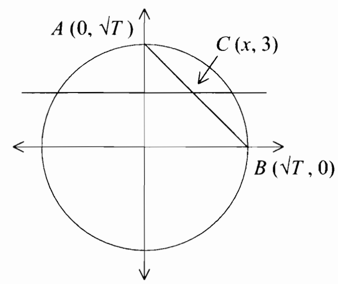
R1–1. Treat Mary, Bob, and Jane as a single bloc and imagine that there are 4 places to sit. Mary, Bob, and Jane can choose any one of 4 places and Mary and Jane may switch, giving $2 \cdot {}_4C_1 = 8$ possible arrangements. The other students may be seated in $3! = 6$ ways, giving $6 \cdot 8 = \boxed{48}$ different ways to sit.

R1–2. Since $BE = AB$ and $CE = CD$, then $AB + CD = BC = 18$. The area of trapezoid $ABCD$ is $\frac{1}{2}(AD)(AB + CD) = \frac{1}{2} \cdot \frac{T}{3} \cdot 18 = 3T$. Since $T = 48$, the answer is $3 \cdot 48 = \boxed{144}$.



R1–3. The equation of $\overleftrightarrow{AB}$ is $x + y = \sqrt{T}$ and since $y = 3$ we have $C\left(\sqrt{T} - 3, 3\right)$. Thus, $AC = \sqrt{\left(\sqrt{T} - 3\right)^2 + \left(3 - \sqrt{T}\right)^2} = \left|\sqrt{T} - 3\right|\sqrt{2}$ and $BC = \sqrt{\left(\left(\sqrt{T} - 3\right) - \sqrt{T}\right)^2 + (3 - 0)^2} = 3\sqrt{2}$.

Since $T = 144$, then AC is the longer and $AC = \boxed{9\sqrt{2}}$.



Solutions to the ARML Relay #2 – 2003

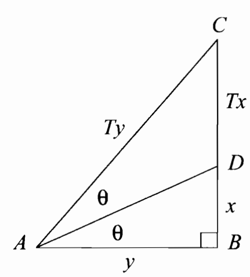
R2–1. The distance from A to B is simply the diagonal of a box whose sides are $14 - 5 = 9$, $14 - 8 = 6$, and $14 - 12 = 2$. The diagonal has the same length no matter how the smaller box is oriented. That length is $\sqrt{2^2 + 6^2 + 9^2} = \sqrt{121} = \boxed{11}$.

R2–2. Since $\left(\log_K 16\right)^2 = x\left(\log_{16} K\right) = \dfrac{x}{\log_K 16}$, then $x = \left(\log_K 16\right)^3$. Since $K = 4$, then $x = \boxed{8}$.

R2–3. By the Triangle Angle Bisector Theorem, since $\dfrac{CD}{DB} = T$, then $\dfrac{AC}{AB} = T$.

Thus, $\cos 2\theta = \dfrac{y}{Ty} = \dfrac{1}{T} \rightarrow 2\cos^2\theta - 1 = \dfrac{1}{T} \rightarrow \cos^2\theta = \dfrac{T+1}{2T}$.

Since $T = 8$, $\cos^2\theta = \boxed{\dfrac{9}{16}}$.



Note: Pass from position 1 to 8 and from position 15 to 8.

1. Let a, b, and c be positive integers such that $a + b + c = 7$. Compute the smallest possible value of $a^2 + b^2 + c^2$.

2. Let T = TNYWR. A farmer has T animals. Some are horses, the rest are chickens. There are a total of 46 legs. Compute the number of horses.

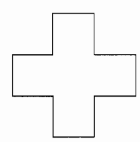
3. Let T = TNYWR. P and Q are right circular cones with radii of a and b respectively and altitudes of m and n respectively. The volume of Q is T times that of P and $b = \frac{a}{2}$. Compute $\frac{n}{m}$.

4. Let T = TNYWR. Let $u(x)$ denote the units digit of the integer x. Compute the value of

$$u\left(7^{T+2003}\right) + u\left(9^{T+2003}\right) + u\left(9^{T+2002}\right) + u\left(7^{T+2001}\right).$$

5. Let T = TNYWR. Let $K = T + 1$. Consider the number $N = \left(2^x\right)\left(3^y\right)$ where x and y are non-negative integers. Compute the smallest value of N such that N has exactly K positive factors.

6. Let T = TNYWR. In the figure all sides are equal and the angle between each pair of consecutive sides is 90°. If the numerical value of the figure's area exceeds its perimeter by T, compute the perimeter of the figure.



7. Let T = TNYWR. If $\log_2 T = a + b\log_2 3$ and $\log_6 T = b + b\log_6 2$, compute $a + b$.

8. Let a and b be the two integers you will receive and let $K = |a - b|$. For x in radians, the graphs of $y = \frac{x}{K}$ and $y = \left|\sin(K\pi x)\right|$ intersect in N points. Compute N.

15. A point (a, b) is chosen at random from the region bounded by the coordinate axes and the lines $x = 10$ and $y = 10$. Compute the probability that $a + b \leq 8$.

14. Let T = TNYWR and let K be the sum of the numerator and denominator of T when T is written in lowest terms. Compute the solution to $\sqrt{x + K} = 11 - \sqrt{x}$.

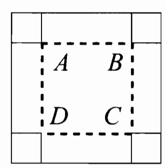
13. Let T = TNYWR and let $K = T - 13$. The radius of a cylinder is K and the volume of the cylinder is numerically equal to the sum of its lateral area and the area of the two bases. Compute the cylinder's height.

12. Let T = TNYWR. If $\dfrac{10T - 7 + i}{T + 2i}$ is written as $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$, compute a.

11. Let T = TNYWR. Compute the smallest integer solution to $x^2 + Tx + T = 1$.

10. Let T = TNYWR and let $K = T + 246$. If K is the first term of an arithmetic sequence and 2003 is the tenth term, compute the common difference.

9. Let T = TNYWR. The large square has an area of T. Each of the small squares shares a vertex with the large square and has sides of length of 4. Compute the area of square $ABCD$.



8. Let a and b be the two integers you will receive and let $K = |a - b|$. For x in radians, the graphs of $y = \dfrac{x}{K}$ and $y = |\sin(K\pi x)|$ intersect in N points. Compute N.

1. 17
2. 6
3. 24
4. 20
5. 576
6. 144
7. 6

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15. $\frac{8}{25}$

14. 16

13. 6

12. 8

11. –7

10. 196

9. 36

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8. 1800

1. Let $a \le b \le c$. There are only 4 triples satisfying $a + b + c = 7$, namely (1, 1, 5), (1, 2, 4), (1, 3, 3), and (2, 2, 3). The smallest occurs when $(a, b, c) = (2, 2, 3)$ and that triple gives $a^2 + b^2 + c^2 = \boxed{17}$.

2. Since $h + c = T$ and $4h + 2c = 46$, we obtain $h = 23 - T$. Since $T = 17$, then $h = \boxed{6}$.

3. From $\frac{1}{3}\pi b^2 n = T\left(\frac{1}{3}\right)\pi a^2 m$ and $b = \frac{a}{2}$, we obtain $\left(\frac{a}{2}\right)^2 n = Ta^2 m \rightarrow n = 4Tm \rightarrow \frac{n}{m} = 4T$.

   Since $T = 6$, then $\frac{n}{m} = \boxed{24}$.

4. If n is a non-negative integer, then $u\left(9^n\right)$ is 1 if n is even and 9 if n is odd. Thus, $u\left(9^{n+1}\right) + u\left(9^n\right) =$ $1 + 9$ or $9 + 1 = 10$. Similarly, $u\left(7^n\right)$ is 1 if $n = 4k$, 7 if $n = 4k + 1$, 9 if $n = 4k + 2$, and 3 if $n = 4k + 3$.

   Thus, $u\left(7^n\right) + u\left(7^{n+2}\right) = 1 + 9,\ 9 + 1,\ 7 + 3$, or $3 + 7 = 10$. The sum is $\boxed{20}$ and does not require T.

5. $N = 2^x \cdot 3^y$ has $(x + 1)(y + 1) = K = 21$ factors. Thus, $(x, y) = (0, 20), (2, 6), (6, 2)$, and $(20, 0)$.

   $N = 2^x \cdot 3^y$ is least when $(x, y) = (6,2) \rightarrow N = \boxed{576}$.

6. If x is the side of the figure, then the area equals $5x^2$ and the perimeter equals $12x$ giving $5x^2 - 12x - T = 0$.

   Since $T = 576$, then $5x^2 - 12x - 576 = 0 \rightarrow (5x + 48)(x - 12) = 0 \rightarrow x = 12 \rightarrow$ perimeter $= \boxed{144}$.

7. $T = 2^{a + \log_2 3^b} = 2^a \cdot 3^b$. Also, $T = 6^{b + \log_6 2^b} = 6^b \cdot 2^b = 2^b \cdot 3^b \cdot 2^b$. From $2^a \cdot 3^b = 2^{2b} \cdot 3^b$, we obtain $a = 2b \rightarrow a + b = 3b$. Since $T = 144 = 2^{2b} \cdot 3^b = 16 \cdot 9$, we obtain $b = 2$ making $3b = \boxed{6}$.

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15. The area under $x + y = 8$ is $\frac{1}{2}(8)(8) = 32$. The probability equals $\frac{32}{100} = \boxed{\frac{8}{25}}$.

14. Squaring yields $x + K = 121 - 22\sqrt{x} + x$. Isolate and square obtaining $x = \left(\frac{121 - K}{22}\right)^2 = \left(\frac{88}{22}\right)^2 = \boxed{16}$.

13. $\pi K^2 h = 2\pi K^2 + 2\pi K h \rightarrow h = \frac{2K}{K - 2}$. Assuming integer values we find $(K, h) = (3, 6), (4, 4)$, and $(6, 3)$ making $h = 3, 4$, or 6. Since $K = 3$, then $h = \boxed{6}$.

12. $\frac{10T - 7 + i}{T + 2i} \cdot \frac{T - 2i}{T - 2i} = \frac{\left(10T^2 - 7T + 2\right) + \left(14 - 19T\right)i}{T^2 + 4}$. Thus, $a = \frac{10T^2 - 7T + 2}{T^2 + 4}$. Since $T = 6, a = \boxed{8}$.

11. $x = \frac{-T \pm \sqrt{T^2 - 4(T - 1)}}{2} = \frac{-T \pm \left|T - 2\right|}{2} = -1$ or $1 - T$ whether $T \geq 2$ or $T < 2$. Since $T = 8, x = \boxed{-7}$.

10. $K = -7 + 246 = 239$. Let d be the common difference $\rightarrow$ $2003 = K + 9d = 239 + 9d \rightarrow d = \boxed{196}$.

9. Large square's side $= \sqrt{T} \rightarrow$ area of $ABCD = \left(\sqrt{T} - 2 \cdot 4\right)^2 = (14 - 8)^2 = \boxed{36}$ since $T = 196$.

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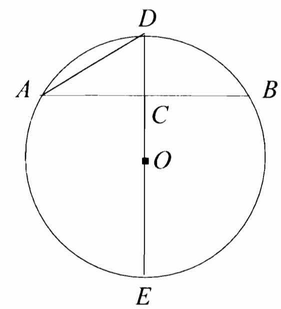
8. The period of $y = \left|\sin K\pi x\right|$ is $\frac{\pi}{K\pi} = \frac{1}{K}$. If the graphs intersect at (m, n), then $n \leq 1 \rightarrow \frac{m}{K} \leq 1 \rightarrow$ $m \leq K$. Note that the line intersects the curve twice in each period. There are $\frac{K}{1/K} = K^2$ periods so there are $2K^2$ points of intersection. Since the answer from #7 is 6 and the answer from #9 is 36, then $K = \left|36 - 6\right| = 30$. Thus, the number of points of intersection is $2 \cdot 30^2 = \boxed{1800}$.

1. Point D lies on circle O such that $\overline{OD} \perp \overline{AB}$. If $AD = 40$, DC is an integer and $DC < OD$, compute the number of values of DC such that the circle's diameter is an integer.



2. Let $[\,x\,]$ = the greatest integer $\leq x$. Compute the least solution to $\dfrac{x}{[x]} = \dfrac{2002}{2003}$. Write your answer in the form $\dfrac{a}{b}$ where a and b have no common factors.

3. Let N be an integer such that the product $41 \cdot 43 \cdot N$ can be written as the sum of 6 consecutive positive integers. Compute the least value for the smallest of the 6 integers.

1. $AC^2 = DC \cdot CE \rightarrow CE = \dfrac{AC^2}{DC}$. $DE = DC + \dfrac{AC^2}{DC} = \dfrac{DC^2 + AC^2}{DC} = \dfrac{AD^2}{DC} = \dfrac{40^2}{DC} = \dfrac{2^6 5^2}{DC}$.

There are $7 \cdot 3 = 21$ divisors of 40^2 and we seek all those divisors DC where $DC < \dfrac{1}{2}DE \rightarrow$

$DC < \dfrac{1}{2} \cdot \dfrac{40^2}{DC} \rightarrow DC < 20\sqrt{2} \approx 28.28$. Thus, DC can be a factor of 40^2 as large as 28, and so

$DC = 1, 2, 4, 5, 8, 10, 16, 20, 25$. Answer: $\boxed{9}$.

2. Let $x = n + h$ where n is an integer and $0 \le h < 1$. Then $\dfrac{n+h}{n} = \dfrac{2002}{2003} \rightarrow h = -\dfrac{1}{2003}n$. Thus,

$-2002 \le n < -1$. Choose $n = -2002$, making $h = \dfrac{2002}{2003} \rightarrow x = -2002 + \dfrac{2002}{2003} = \dfrac{2002(1-2003)}{2003} =$

$\boxed{\dfrac{-4{,}008{,}004}{2003}}$.

3. Let x be the smallest of the six integers. Then $x + (x+1) + \ldots + (x+5) = 6x + 15 = 3(2x+5) =$

$41 \cdot 43 \cdot N$. So N is divisible by 3. If N were 3, then $2x + 5 = 41 \cdot 43 = 1763 \rightarrow x = \dfrac{1758}{2} = \boxed{879}$.

Rook Polynomials

Induction

Regular Closed Linkages

Slides, Rolls, and Rolides

Twenty-five Point Affine Geometry

Unit Fractions

Cevians

Integer Geometry

Errors in Mathematical Reasoning

ARML Power Contest

1994 - 2004

STRINGS

Pythagorean Triples

Three Addition Problems

Color Transformations

Factorial Polynomials

Square - Sum Partitions

Insane Tic-Tac-Toe

Number Theoretic Functions

Chromatic Polynomials

Rotating Decimals

Color Transformations

The Definitions and Notation

Begin with a regular grid composed of squares, each colored black or white. For example:

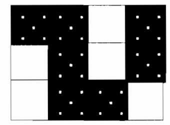


Figure 1

We begin changing this color according to the following rule: In one step we may either reverse the colors of all the squares in one row or reverse all the colors in one column. So for example, starting with Figure 1, in one step we could get the coloring in Figure 2 by reversing row 2, or we could get the coloring in Figure 3 by reversing column 3. If we took several steps, the resulting coloring might look quite different form Figure 1. See Figure 4, in which we've reversed row 2, then column 2, then column 4.

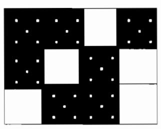


Figure 2

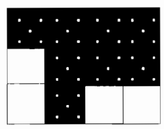


Figure 3

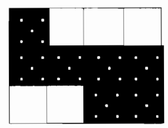


Figure 4

We use the notation <u>r*i*</u> to signify the act of flipping or reversing the colors in row *i* and <u>c*i*</u> will refer to flipping or reversing the colors in column *i*. We will record a sequence of flips in a <u>flip string</u>. So the flip strings representing Figures 2, 3, and 4 above would be r2, c3, and r2-c2-c4.

One coloring will be called <u>accessible</u> from another if either coloring can be transformed into the other by a sequence of flips.

The Problems

1. Compute the results of applying the following flip strings to the coloring of Figure 1, by drawing the resulting colorings on your answer sheet.
   1a. r1-c1-r2-c4   1b. r2-r1-c4-c1   1c. r3-c2-c3   1d. r1-r1-r1-c2-r3-c2-r1
   1e. r3

2. Construct two different flip strings each of which turns Figure 1 into the Figure 5.

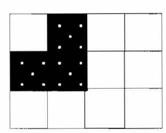


Figure 5

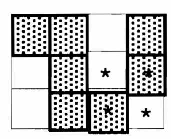


Figure 6

3. Starting with Figure 1, construct a flip string which turns the four squares in Figure 6 marked with an asterisk all white.

4. Starting with Figure 1, construct a flip string which turns the four squares in Figure 7 marked with an asterisk all white.

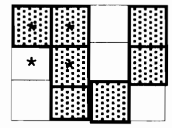


Figure 7

5. In a m by n grid, how many colorings (different colored grids) have the bottom row and the right-hand column entirely white?

6. Can every coloring be transformed into such a coloring? Why or why not?

7. In a 2 by 2 grid, if the initial coloring contains exactly one black square, how many colorings are accessible from it? (Proof not required.)

8. In a 2 by 2 grid, if the initial coloring contains exactly two black squares, how many colorings are accessible from it? (Proof not required.)

9. Show that if we start with any five different colorings of a 2 by 3 grid, we can find two colorings among them, each of which is accessible from the other.

10. Suppose you start with a grid which is larger than a 2 by 2 and has only one black square. Can you ever get a coloring that is all black? Why or why not?

Given a coloring, we can try to eliminate as many blacks as possible, seeking a coloring with a minimum number of black squares. We call this number the <u>minimum</u> of the coloring. For example, the coloring in Figure 8 has a minimum of 0, since we can eliminate all the black squares by c1-c2-c4.

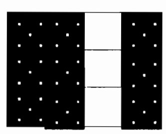


Figure 8

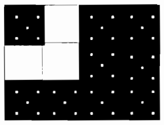
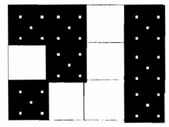
11. Show that all colorings accessible from one another have the same minimum.

12. A given coloring, together with all the those accessible from it, form a coloring class. As a result of problem 12, we can speak of the <u>minimum of a class</u>. What values occur as the minimum of a class for the following grid sizes:

12a. 2 by 3   12b. 4 by 4   12c. m by n

==

The Solutions

1a., 1b., 1c.  1d., 1e. 

2. Two solutions are: c2-c3-r1-r2 and r1-r2-c2-c3. Many other examples are possible.

3. One possible answer is r2-c3.

4. The only flips that affect these squares are r1, r2, c1 and c2. In the present state, r1 will change two black squares to two white squares (or zero black squares). If the two top right squares are both white, then r1 will turn them both black. If one of these squares is white and the other black, r1 simply reverses their colorings. In all cases, r1 changes the number of black squares among the top four by an even number, so that the number of black squares remains odd. It is not difficult to see that the same is true of r2, c1, and c2. Hence any application of these flips, in any order, will result in an odd number of black squares among the top right four. Since 0 is an even number, no flip string can ever result in 0 black squares and four white squares.

5. In a *m* by *n* grid, there are 2^{mn} colorings, since each of these *mn* squares may be colored either black or white. The number of colorings described in the problem is only $2^{(m-1)(n-1)}$ since there is no choice for the nth row or the mth column.

6. The following algorithm will turn any coloring into a coloring with the bottom row and right-hand column entirely white: First examine the bottom square of every column. If that square is black, flip the entire column. After going through each column, the entire bottom row will be white. Now examine every row *except* the bottom one. If the rightmost square in the row is black, flip the entire row. After going through each row (except the bottom one), the right column of the grid will be all white.

7. and 8. Answer: 8 in each case. Following the line of reasoning described in #4 above, it becomes clear that for a 2 by 2 grid, the parity (evenness or oddness) of the number of black squares does not change. Therefore a grid with a single black square can only be transformed into one with oddly many black squares. A quick count shows that there are eight of these: four with one black square and four with three. Similarly, a grid with two black squares can only be transformed into one with evenly many black squares. These are the other 8 possible colorings.

For a full proof, it would remain to show that each of the 8 odd colorings can be obtained from any other, as can each of the 8 even colorings. This task can be accomplished by actual construction.

9. We will show there are only four "classes" of colorings of a 2 by 3 grid. Any two colorings within one class are

mutually accessible, and any two colorings in different classes are mutually inaccessible. Therefore, if we have five colorings, at least two must belong to the same class, and so are accessible from each other.

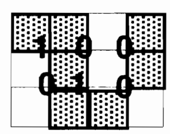
First look at the first two columns of the 2 by 3 grid, and consider these four squares as a 2 by 2 grid in its own right. We have seen in #7 and 8 that there are only two kinds of these grids: those with evenly many black squares and those with oddly many. A 2 by 3 grid with evenly many black squares in the first two columns can be transformed into one with no black squares in the first column, and a 2 by 3 grid with oddly many black squares in the first two columns can be transformed into one with a single black square, say, in the upper right hand position. Furthermore, no 2 by 3 grid of the first class can be transformed into one of the second class.

Having transformed any 2 by 3 coloring as described above, we now look at its last two columns. Again, they form a 2 by 2 grid in its own right, and so there are two classes it can belong to, according as the number of black squares is even or odd (if it has evenly many black squares, flips can make it all white, and if it has oddly many black squares, flips can give it a single black square, say in the upper center of the original grid).

Hence we can classify any coloring in two ways according to its first two columns, and in two ways according to its last two columns. It is not difficult to see that these classifications result in four classes: odd or even for the first two columns, and odd or even with the last two. Together with out initial observation, this proves the assertion.

10. We have seen (questions #6 and #7) that a 2 by 2 grid with one black square cannot be turned entirely white. We can extend this to any grid by considering the 2 by 2 "subgrid" which includes the single black square. Considering this as a 2 by 2 grid in its own right, we see that if it has a single black square to begin with, it must have oddly many black squares after a sequence of flips. The all-black coloring has evenly many black squares (four of them) in this grid, and so is not accessible from the initial coloring.

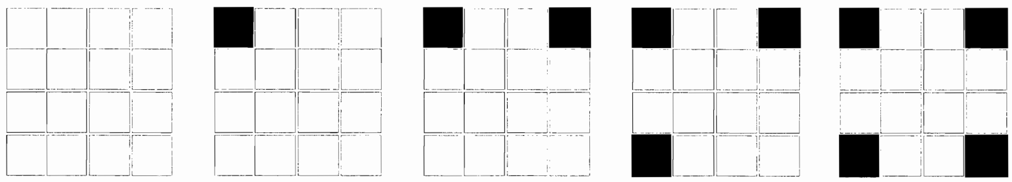
11. If coloring Y can be transformed into coloring Z with k black squares, then any other coloring X accessible from Y can also be transformed into Z, for example, by combining the flip strings effecting the transformations. Thus the minimum of Y is the same as the minimum of any other coloring assessable from Y.



12. To solve this set of problems, we introduce the notion of a *parity lattice*. We have already seen that if a grid has more than two rows or columns, then any 2 by 2 subgrid always has either evenly many or oddly many black squares, no matter how we transform the coloring. Thus we can affix a 0 or a 1 to every intersection of four squares, according as there are evenly many (indicated by 0) or oddly many (indicated by 1) black squares among the four that share the corner. We call this array, derived from any coloring, its *parity lattice*. An example, based on the coloring of figure 1, is shown above on the left. What we have just shown is that the parity lattice of a coloring does not change as its rows or columns are flipped.

12a. The possible minimum values are 0 and 1. A 2 by 3 grid has a 1 by 2 parity lattice, and there are four such: 00, 01, 10, 11. Thus there are four classes of 2 by 3 colorings. A set of minimal colorings for the classes was given in the answer to #9.

12b. The possible minima for 4 by 4 colorings are 0, 1, 2, 3, and 4. Examples of colorings having these minima are:



An argument showing that these colorings do in fact have the stated minima, and that no other minima are possible, follows from the following theorem:

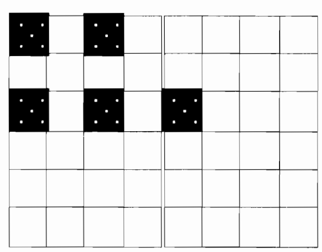
Theorem: For an m by n grid with $m = 2p$, $n = 2q$, there is a class with S as its minimum for each $0 \leq S \leq pq$.

Proof: Rather than viewing the grid as an m by n table of squares, view it as a p by q table of non-overlapping 2 by 2 subgrids. Since the parity lattice of the table doesn't change when flips are applied, a 2 by 2 subgrid containing an odd number of black squares will *always* contain an odd number of black squares, no matter how the coloring is transformed. In particular, it must always contain at least one black square. So if U and V are the 2 by 2 subgrids below:

subgrid U subgrid V

Then a coloring composed of T copies of U and pq-T copies of V will have a minimum of T.

A 6 by 8 coloring with a minimum of 5:



Observe that it is impossible for a class to have a minimum which is greater than pq, since any coloring of a $2p$ by $2q$ grid with at least $pq + 1$ black squares must have some row or some column which is more than half black (this is not obvious, but can be shown, for example, by using the pigeon hole principle). By flipping this row or column you reduce the number of black squares in the coloring. So no coloring with $pq + 1$ or more black squares can be a minimum coloring of its class, and pq is in fact the "maximal minimum."

Induction

The Problems

1. Suppose you are given a square and asked to subdivide it into n non-overlapping squares. Figure 1 shows that it can be done if n = 4. (It is impossible if $n = 5$ or $n < 4$.)

   a) Show that a square can be subdivided into n squares if $n = 6$.

   b) Show that a square can be subdivided into n squares if $n = 7$.

   c) Show that a square can be subdivided into n squares if $n = 8$.

   d) Prove that a square can be subdivided into n squares if $n > 8$.

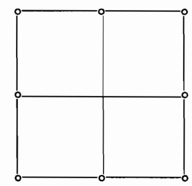


Figure 1

2. Suppose you are given an equilateral triangle and asked to subdivide it into n non-overlapping equilateral triangles. Figure 2 shows that it can be done if $n = 4$. (It is impossible if $n = 5$ or $n < 4$.)

   a) Show that an equilateral triangle can be subdivided into n non-overlapping equilateral triangles if $n = 6$.

   b) Show that an equilateral triangle can be subdivided into n non-overlapping equilateral triangles if $n = 7$.

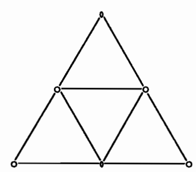


Figure 2

   c) Show that an equilateral triangle can be subdivided into n non-overlapping equilateral triangles if $n = 8$.

   d) Prove that an equilateral triangle can be subdivided into n non-overlapping equilateral triangles if $n > 8$.

3. Suppose you are asked to draw a three-dimensional solid figure with n edges. Figure 3 shows that this can be done if $n = 6$. (This is impossible to do if $n = 7$ or $n < 6$.)

   a) Make a sketch of a three-dimensional figure with n edges if $n = 8$.

   b) Make a sketch of a three-dimensional figure with n edges if $n = 9$.

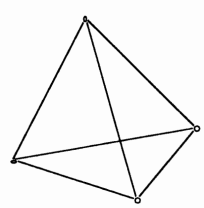


Figure 3

c) Make a sketch of a three-dimensional figure with n edges if $n = 10$.

d) Prove that a three-dimensional figure with n edges can always be made if $n > 10$.

4. Suppose you have only 3¢ and 5¢ stamps. Using a 3¢ and a 5¢ stamp, you could come up with postage for 8¢. Using three 3¢ stamps, you could come up with postage for 9¢. Using two 5¢ stamps, you could come up with postage for 10¢. It would be impossible to come up with postage for 7¢.

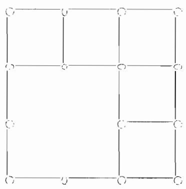
   Prove it is possible, using only 3¢ and 5¢ stamps, to come up with all postages greater than or equal to 8¢.

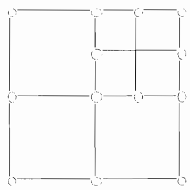
5. Suppose you have only 3¢ and X¢ stamps (where X is not a multiple of 3). Find an expression for the largest amount of postage which you cannot make using the 3¢ and X¢ stamps.

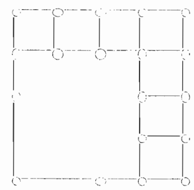
6. Suppose you have only Y¢ and X¢ stamps (where Y and X are relatively prime). Find an expression for the largest amount of postage which you cannot make using the Y¢ and X¢ stamps.

7. Suppose you have only Y¢ and X¢ stamps (where Y and X are relatively prime) and w equals the largest amount of postage that you cannot make using the Y¢ and X¢ stamps, prove that of all (positive) postage amounts, a and b, where $a + b = w$, either a or b but not both can be made using Y¢ and X¢ stamps.

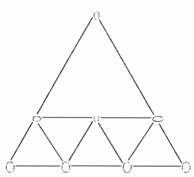
The Solutions

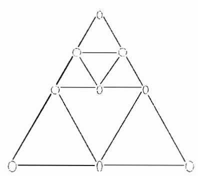
1a) 

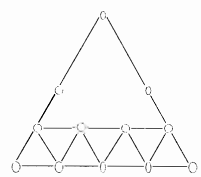
b) 

c) 

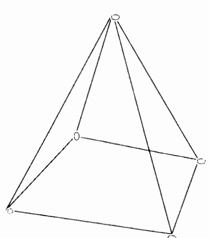
d) Any subsquare in any of the above three diagrams can be subdivided into 4 squares as shown in Figure 1. The net result would be an increase of three in the total number of subsquares in each figure. Any number $n > 8$ can be written in one of the forms $3m + 6$, $3m + 7$, or $3m + 8$, where m is some positive integer. Each of these forms determines one of the three figures above *a, b,* and *c* respectively and m would be the number of subsquares that must be subdivided into four squares to make the total number of subsquares equal to n.

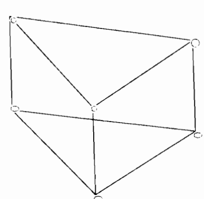
2a) 

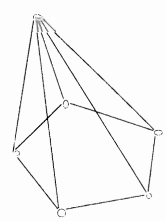
b) 

c) 

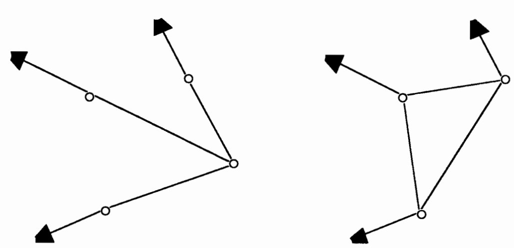
d) Any subtriangle in any of the above three diagrams can be subdivided into 4 triangles as shown in Figure 2. The net result would be an increase of three in the total number of subtriangles in each figure. Any number $n > 8$ can be written in one of the forms $3m + 6$, $3m + 7$, or $3m + 8$, where m is some positive integer. Each of these forms determines one of the three figures above, *a, b,* and *c* respectively, and m would be the number of subtriangles that must be subdivided into four triangles to make the total number of subtriangles equal to n.

3a) 

b) 

c) 

d) Each of the above figures contains a vertex of order 3, i.e. a vertex connected to three edges. As shown in the diagram below, any vertex of order 3 can be truncated or cut off and the three "dangling" edges connected with three edges. The net result is that the figure now has three more vertices of degree 3 and three more edges.



This truncating process can continue and will always add three more edges onto the figure. Any number $n > 10$ can be written in one of the forms $3m + 8$, $3m + 9$, or $3m + 10$, where m is some positive integer. Each of these forms determines one of the three figures above, a, b, and c respectively, and m would be the number of vertices of order 3 that need to be truncated so that the figure has n edges.

4. It has already been shown that 3¢ and 5¢ stamps can be used for 8¢, 9¢, and 10¢ postage. Any number $n > 10$ can be written in one of the forms $3m + 8$, $3m + 9$, or $3m + 10$, where m is some positive integer. Postage in the form $3m + 8$ can be made with a five-cent stamp and $m + 1$ three-cent stamps. Postage in the form $3m + 9$ can be made with $m + 3$ three-cent stamps. Postage in the form $3m + 10$ can be made with two five-cent stamps and m three-cent stamps.

5. Let f(X) = the largest amount of postage that cannot be made with 3¢ and X ¢ stamps. Experiment with various values of X to complete this table:

X	4	5	7	8	10 ...
$f(X)$	5	7	11	13	17 ...

It can be shown that $f(X) = 2X - 3 = 2(X - 1) - 1$.

6. Let $f(X, Y)$ = the largest amount of postage that cannot be made with Y ¢ and X ¢ stamps. Experimentation again shows $f(X, Y) = (X - 1)(Y - 1) - 1$ or $f(X, Y) = XY - (X + Y)$.

X

Y	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2		1		3		5		7	
3	1		5	7		11	13		17
4		5		11		17		23	
5	3	7	11		19	23	27	31	
6				19		29			
7	5	11	17	23	29		41	47	53
8		13		27		41		55	
9	7		23	31		47	55		71
10		17			44	53		71	
11	9	19	29	39	49	59	69	79	89
12				43		65			
13	11	23	35	47	59	71	83	95	107

$f(X, Y)$

For given X and Y, call an integer $n \geq 0$ *postable* if $n = aX + bY$ for some nonnegative integers a and b.

Then $XY - (X + Y)$ is not postable: Assume $XY - (X + Y) = aX + bY$. Then $XY = (a + 1)X + (b + 1)Y$. Notice that if a and b are both nonnegative, this implies that $(a + 1)X < XY$ so that $a + 1 < Y$ and $a < Y - 1$. Similarly, $b < X - 1$. But then, $(b + 1)\ Y = (Y - a - 1)\ X$, implying that $Y \mid (Y - a - 1)$ because Y and X are relatively prime. But obviously, $0 < Y - a - 1 < Y$, so this is impossible.

Proving that any larger number is postable is trickier; it's enough to prove that the next Y integers are postable, since any larger integer can then be written by adding a multiple of Y. Since X and Y are relatively prime, for any k in $\{1, 2, \ldots, Y - 1\}$ there exist nonnegative integers a and b such that $aX = bY + k$. Clearly $a > 0$, but also remember that we can assume $a < Y$ and $b < X$, because if $a > Y$ then $b > X$ and $(a - Y)X = (b - X)Y + 1$, we can continue until $a < X$ and $b < Y$. Then $aX - bY = k$, where $b \leq X - 1$, so $aX + (X - 1 - b)Y = XY + k - Y$; since $a \geq 1$ we then take $(a - 1)X + (X - 1 - b)Y = XY - X - Y + k$ with both $a - 1$ and $X - 1 - b$ nonnegative. For $k = Y$, we get $XY - X$, which is a multiple of X.

7. First note that $(X - 1)(Y - 1)$ is even: X and Y are relatively prime, so at least one of them is odd, meaning that at least one of $X - 1$ and $Y - 1$ is even. We now show that the number of postable integers n in $\{0, 1, \ldots, XY - X - Y\}$ is exactly equal to $(X - 1)(Y - 1) / 2$.

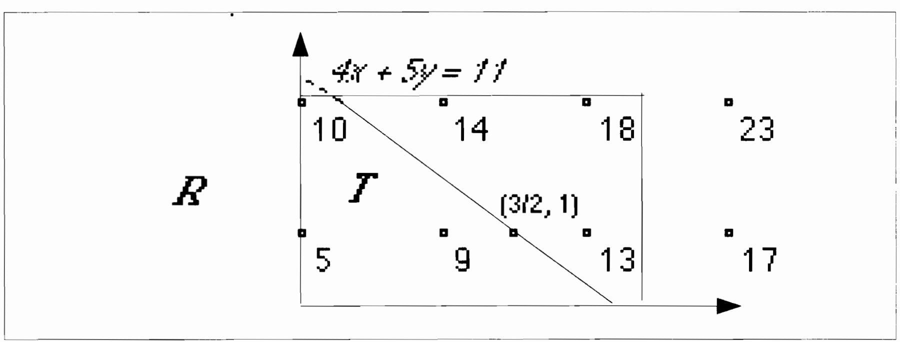
Let's define three distinct but related sets:

Let N be the set of postable integers n in $\{0, 1, \ldots, XY - X - Y\}$.

Let T (for triangle -- see figure) be the set of pairs of nonnegative integers (a, b) such that $aX + bY < (X - 1)(Y - 1)$.

Let R (for rectangle) be the set of pairs of nonnegative integers (a, b) such that $a < Y - 1$ and $b < X - 1$.

The diagram shows the case $X = 4$, $Y = 5$.



Notice two fundamental facts. First: T is a subset of R, because if (a, b) is in T then $aX < aX + bY < (X - 1)(Y - 1) < X(Y - 1)$, so $a < Y - 1$; likewise $b < X - 1$. Second, every element of R defines a unique sum $aX + bY$, because if $aX + bY = a'X + b'Y$ then $(a - a')X = (b - b')Y$ so that $Y \mid (a - a')$ and $X \mid (b - b')$; this is impossible if a and a' (also b and b') are in the appropriate ranges. It follows from this that N has exactly the same number of elements as T, since no element of N can correspond to more than one element of T.

Now we show that exactly half the elements of R are in T.

Let $r(a, b) = (Y - 2 - a, X - 2 - b)$. This is a 180 degree rotation about the point $(Y / 2 - 1, X / 2 - 1)$ which, we will show, maps T to R – T and vice versa.

First, $r(a, b)$ is in R for any (a, b) in R. Since at least one of $X - 2$ and $Y - 2$ is odd, no element maps to itself; but $r(r(a, b)) = (a, b)$, so r breaks up R into doublets of points $\{(a, b), r(a, b)\}$. In each doublet, the two sums together add up to exactly $2(XY - X - Y)$:

$$(aX + bY) + (Y - 2 - a)X + (X - 2 - b)Y = (Y - 2)X + (X - 2)Y = 2(XY - X - Y).$$

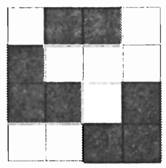
No element has the sum $XY - X - Y$, so one element of each doublet has a sum less than $XY - X - Y$, the other has a sum greater than $XY - X - Y$. Therefore each doublet contains exactly two distinct elements of R with no overlap between the doublets, and the union of the doublets is R; so the number of doublets must be half the number of element of R, or $(X - 1)(Y - 1) / 2$. For every doublet there is one element of T, and for every element of T there is one element of N, so the number of elements of N is $(X - 1)(Y - 1)/2$.

However, because $XY - X - Y$ is not postable, it is not the sum of any two postable integers either. Therefore, if we divide all numbers in $\{0, 1, ..., XY - X - Y\}$ into pairs (m, n) with $m + n = XY - X - Y$, each pair has at most one postable element. Again, $XY - X - Y$ is odd, so each pair has two distinct numbers. But if any pair did not have at least one postable number, the total number of postables would be less than $(X - 1)(Y - 1)/2$, since there are $(X - 1)(Y - 1) / 2$ pairs and none of them has two postables! Therefore in every case, exactly one of n and $XY - X - Y - n$ is postable.

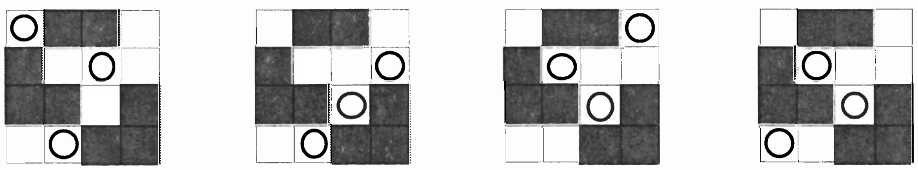
Rook Polynomials

The Definitions

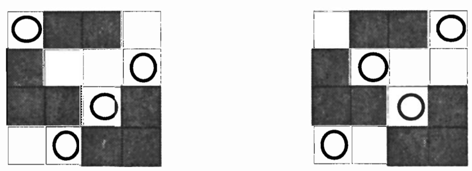
A **board** consists of an *n* by *m* array of squares, some of which are unshaded (available) and others are shaded (unavailable). **Rooks** (markers) can be placed on any available square provided that it is **non-challenged**, that is, no two rooks are in the same row or column.



In the above board, a single rook can be placed in eight different ways into the eight available squares. However, there are nineteen different ways to place two rooks onto the above board and only fourteen ways to place three rooks onto the board. Some of these fourteen ways are illustrated below:



In addition, as shown below, there are only two different ways to place four rooks on the board above so that no rook is challenging another.



For every board, B, there exists a **rook polynomial**, $R_B(x)$,

where $R_B(x) = 1 + r_1x + r_2x^2 + r_3x^3 + \ldots + r_kx^k + \ldots$, and r_k , the coefficient of x^k , is the number of ways of putting *k* non-challenged rooks on board B. The rook polynomial for the board above is :
$R_B(x) = 1 + 8x + 19x^2 + 14x^3 + 2x^4$. (The first term is one because it is the coefficient of x^0 and there is one way of putting no rooks on any board.)

The following theorems may be helpful in determining and checking rook polynomials for more difficult boards:

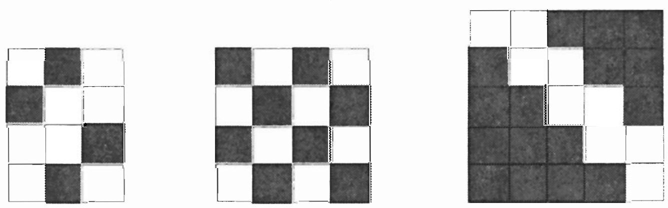
Theorem 1. If board B can be partitioned vertically into two boards C and D where the corresponding rows of C and D do not both contain available squares or if board B can be partitioned horizontally into two boards C and D where the corresponding columns of C and D do not both contain available squares, then $R_B(x) = R_C(x) \cdot R_D(x)$

Theorem 2. Let B be a board and let s be one particular available square in B. Then let B_1 be the board obtained from B by shading in square s and B_2 be the board obtained from B by deleting the row and the column containing s. Then $R_B(x) = R_{B_1}(x) + x \cdot R_{B_2}(x)$

Theorem 3. Let B be an n by n board with rook polynomial $R_B(x) = 1 + r_1x + r_2x^2 + r_3x^3 + \ldots + r_nx^n$ and let ß be the complement of B (ß is shaded where B is unshaded and vice versa). The number of ways of placing n non-challenging rooks on ß is $n! - (n-1)!\,r_1 + (n-2)!\,r_2 - \ldots + (-1)^n(n-n)!\,r_n$.

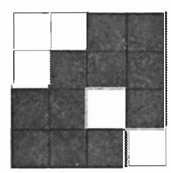
<u>The Problems</u>

1. Determine the rook polynomial for each of the following boards:



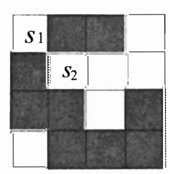
a b c

2a. The rook polynomial for Board 2a is $1 + 5x + 8x^2 + 5x^3 + x^4$. Demonstrate your understanding of Theorem 1 by partitioning the board and factoring this polynomial into two rook polynomials in two different ways.



Board 2a

2b. The rook polynomial for Board 2b is $1 + 7x + 14x^2 + 8x^3 + x^4$. Demonstrate your understanding of Theorem 2 by eliminating square s_1, then s_2 from the resulting two, and writing the polynomial in the form:
$R_B = R_{B_1}(x) + x \cdot R_{B_2}(x) + x \cdot R_{B_3}(x) + x^2 \cdot R_{B_4}(x)$.



Board 2b

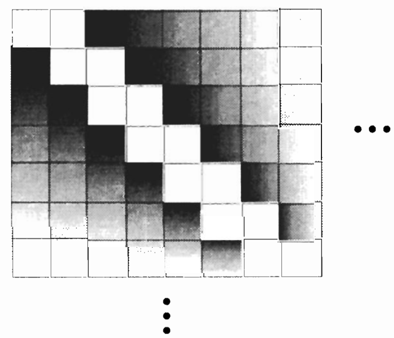
2c. Demonstrate your understanding of Theorem 3 by determining in how many ways five rooks can be placed on a 5 by 5 board whose five downward diagonal squares are the only squares shaded.

2d. Prove Theorem 2: $R_B = R_{B_1}(x) + x \cdot R_{B_2}(x)$.

(Hint: Show that the coefficient of of any x^k is the same on both sides of the equation.)

3a. Find the general rook polynomial for an n by n board where all squares are shaded except those in the downward diagonal.

3b. Write down the general rook polynomial of a n by n board of the type shown, consisting of m available squares (two in each row, going diagonally down the board). All others squares on the board are shaded. (Hint: Think Pascal's Triangle.)



3c. A board is complete if it contains no shaded squares. Find the rook polynomials for complete 1 by 1, 2 by 2, 3 by 3, 4 by 4, 5 by 5, and n by n boards.

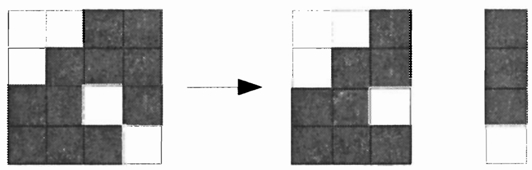
The Solutions

1. a) $1 + 8x + 17x^2 + 8x^3$

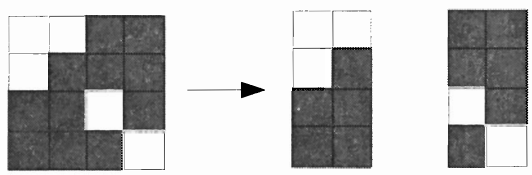
   b) $1 + 8x + 20x^2 + 16x^3 + 4x^4$

   c) $1 + 9x + 28x^2 + 35x^3 + 15x^4 + x^5$

2a. Board 2a can be partitioned in four ways but because of the board's symmetry there are only two partitions that produce different polynomial factors:

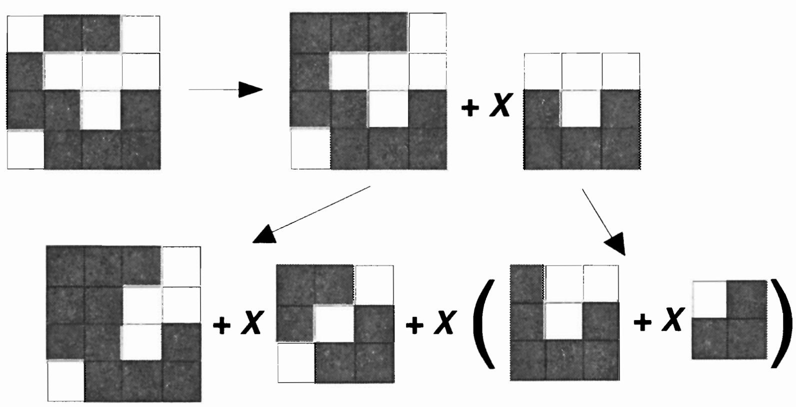


$1 + 5x + 8x^2 + 5x^3 + x^4 = (1 + 4x + 4x^2 + x^3)(1 + x)$



$1 + 5x + 8x^2 + 5x^3 + x^4 = (1 + 3x + x^2)(1 + 2x + x^2)$

2b.

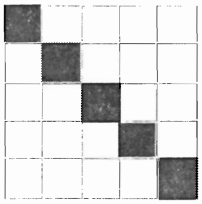


$(1 + 5x + 7x^2 + 3x^3) + x(1 + 3x + 3x^2 + x^3) + x(1 + 3x + x^2) + x^2(1 + x) =$

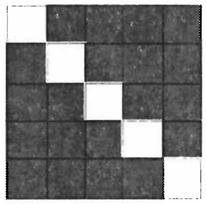
$1 + 5x + 7x^2 + 3x^3 + x + 3x^2 + 3x^3 + x^4 + x + 3x^2 + x^3 + x^2 + x^3 =$

$1 + 7x + 14x^2 + 8x^3 + x^4$

2c.



Board B



Board ß

$R_ß(x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$. Therefore, the number of ways of placing five non-challenging rooks on Board B is $5! - (4!)(5) + (3!)(10) - (2!)(10) + (1!)(5) - (0!)(1) = 44$.

2d. The coefficient of x^k in $R_B(x)$ = the number of ways of placing k rooks on B

= (# of ways of placing k rooks on B with s not used) + (# of ways of placing k rooks on B with s used)

= (# of ways of placing k rooks on B_1) + (# of ways of placing $k - 1$ rooks on B_2)

= (the coefficient of x^k in $R_{B1}(x)$) + (the coefficient of x^{k-1} in $R_{B2}(x)$)

= (the coefficient of x^k in $R_{B1}(x)$) + (the coefficient of x^k in $x \cdot R_{B2}(x)$)

= the coefficient of x^k in $R_{B1}(x) + x \cdot R_{B2}(x)$.

Therefore, $R_B(x) = R_{B1}(x) + x \cdot R_{B2}(x)$.

3a. $R_B(x) = (1 + x)^n$ Using Theorem 1 over and over again $(n - 1)$ times, the board can be partitioned into n boards, each 1 by 1 with a rook polynomial $1 + x$. Therefore, the polynomial for the entire board is $(1 + x)^n$.

3b.

m	$R_B(x)$
1	$1 + 1x$
2	$1 + 2x$
3	$1 + 3x + 1x^2$
4	$1 + 4x + 3x^2$
5	$1 + 5x + 6x^2 + 1x^3$

6	$1 + 6x + 10x^2 + 4x^3$
7	$1 + 7x + 15x^2 + 10x^3 + x^4$
8	$1 + 8x + 21x^2 + 15x^3 + 5x^4$
m	$1 + \binom{m}{1}x + \binom{m-1}{2}x^2 + \binom{m-2}{3}x^3 + \ldots + \binom{m-k+1}{k}x^k + \ldots$

3c.

n	$R_B(x)$
1	$1 + x$
2	$1 + 4x + 2x^2$
3	$1 + 9x + 18x^2 + 6x^3$
4	$1 + 16x + 72x^2 + 96x^3 + 24x^4$
5	$1 + 25x + 200x^2 + 600x^3 + 600x^4 + 120x^5$

Notice that the coefficients of x are the square numbers and the coefficients of x^n are $n!$ (Theorem 3 would prove this last result.) Therefore, the general term will probably contain squares and factorials. With this in mind, the table above can be rewritten:

n	$R_B(x)$
1	$1 + x$
2	$1 + 1(2^2)x + 2(1^2)x^2$
3	$1 + 1(3^2)x + 2(3^2)x^2 + 6(1^2)x^3$
4	$1 + 1(4^2)x + 2(6^2)x^2 + 6(4^2)x^3 + 24(1^2)x^4$
5	$1 + 1(5^2)x + 2(10^2)x^2 + 6(10^2)x^3 + 24(5^2)x^4 + 120(1^2)x^5$

Now the squares and factorials are even more evident and also the binomial coefficients appear.

Therefore, $R_B(x) = 1 + 1!\binom{n}{1}^2 x + 2!\binom{n}{2}^2 x^2 + 3!\binom{n}{3}^2 x^3 + \ldots + n!\binom{n}{n}^2 x^n$

Rotating Decimals

Notation

Throughout this problem we are concerned only with rational numbers that can be represented by totally repeating decimals. For example, $\frac{1}{7} = 0.\overline{142857}$. Partially repeating decimals like $\frac{1}{4} = 0.25\bar{0}$ and $\frac{1}{6} = 0.1\bar{6}$ are not part of this problem. $0.\overline{d_1d_2d_3 \ldots d_n}$, means the totally repeating decimal with digits $d_1, d_2, d_3, \ldots, d_n$ in the repetend. The expression, $\underline{d_1d_2d_3 \ldots d_n}$, means the integer with digits $d_1, d_2, d_3, \ldots, d_n$.

The following is a theorem that you may use without proof:

$$\text{For every totally repeating decimal, } 0.\overline{d_1d_2d_3 \ldots d_n} = \frac{\underline{d_1d_2d_3 \ldots d_n}}{10^n - 1}.$$

For example, $0.\overline{37} = \frac{37}{99}$ and $0.\overline{145} = \frac{145}{999}$.

Definition

For every totally repeating decimal, let's define a rotating function, r, such that

$$r(0.\overline{d_1d_2d_3 \ldots d_{n-1}d_n}) = 0.\overline{d_nd_1d_2 \ldots d_{n-1}}$$

As examples, $r(0.\overline{1234}) = 0.\overline{4123}$ and $r\left(\frac{1}{11}\right) = r(0.\overline{09}) = 0.\overline{90} = \frac{10}{11}$.

The Problems

1a. Compute repeating decimals for $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{3}{7}$ and then compute r ($\frac{1}{7}$), r ($\frac{2}{7}$), and r ($\frac{3}{7}$). Express your final answers as reduced fractions.

1b. Compute repeating decimals for $\frac{1}{13}$ and $\frac{4}{37}$ and then compute r ($\frac{1}{13}$) and r ($\frac{4}{37}$). Express your final answers as reduced fractions.

2a. Prove: For every $x = 0.\overline{d_1d_2d_3 \ldots d_n}$, r $(x) = \frac{x + d_n}{10}$.

2b. Prove: If $\frac{1}{m}$ is a totally repeating decimal, then r ($\frac{1}{m}$) is an integer multiple of $\frac{1}{m}$.

2c. Prove: If $\frac{s}{m}$ is a totally repeating decimal, then $r(\frac{s}{m}) = \frac{t}{m}$ for some integer t.

Give a compact formula for t.

3a. Solve the following two multiplication problems by finding the values of the digits $a, b, c, d,$ and e.

i) $\underline{a\,b\,c\,d\,e\,6} * 4 = \underline{6\,a\,b\,c\,d\,e}$ ii) $\underline{a\,b\,c\,d\,e\,4} * 4 = \underline{4\,a\,b\,c\,d\,e}$

3b. Find a solution to the equation, $r(x) = 4x$. Express your answer as a reduced fraction.

3c. Prove: For each digit L, where $L = 1, 2, 3, \ldots, 9$, there exists an integer n such that the multiplication problem,

$$\overline{d_1\,d_2\,d_3\,\ldots\,d_{n-1}\,L} * L = \overline{L\,d_1\,d_2\,d_3\,\ldots\,d_{n-1}},$$ has a solution.

Consider the set of repeated rotations of $\frac{1}{3^7} = \frac{1}{2187}$, i.e. $r(\frac{1}{2187})$, $r(r(\frac{1}{2187}))$, $r(r(r(\frac{1}{2187})))$, ...

It is a fact (but difficult to prove!) that the repetend of $\frac{1}{2187}$ is 243 digits long! You may use this fact in the following problems.

4a. Prove if $\frac{u}{2187}$ is in the set of rotations of $\frac{1}{2187}$ then $u \equiv 1 \bmod 9$ (i.e. $u = 9k + 1$ for some integer k).

4b. Prove the set of rotations of $\frac{1}{2187}$ is $\{ \frac{1}{2187}, \frac{10}{2187}, \frac{19}{2187}, \ldots, \frac{2179}{2187} \}$.

4c. Prove that each of the 100 pairs of digits 00, 01, 02, ..., 99 appears in the decimal expansion of $\frac{1}{2187}$.

<u>Extensions</u> (not part of the contest problem)

1. When is the decimal equivalent of the fraction $\frac{a}{b}$ (in lowest terms) totally repeating?
2. If gcd(n, 10) = 1 then the length of the repetend of $\frac{1}{n}$ is r, where r is the smallest positive integer such that $10^r \equiv 1 \bmod n$.
3. $\frac{1}{3^t}$ has repetend of length 3^{t-2} for any $t \geq 2$. (A generalization of the fact used in problem #4.)
4. The decimal expansion of $\frac{1}{3^{10}}$ contains every possible three-digit sequence and the decimal expansion of $\frac{1}{3^{500}}$ contains every possible 100-digit sequence.
5. The decimal expansion of the irrational number, $\sum_{n=1}^{\infty} \frac{1}{3^{n^n}}$, contains every digit sequence of every length !!!

The Solutions

1a. $r(1/7) = r(0.\overline{142857}) = 0.\overline{714285} = 5/7$

$r(2/7) = r(0.\overline{285714}) = 0.\overline{428571} = 3/7$

$r(3/7) = r(0.\overline{428571}) = 0.\overline{142857} = 1/7$

1b. $r(1/13) = r(0.\overline{076923}) = 0.\overline{307692} = 4/13$

$r(4/37) = r(0.\overline{108}) = 0.\overline{810} = 30/37$

2a.

$$\frac{x + d_n}{10} = \frac{\frac{d_1 d_2 d_3 \ldots d_n}{10^n - 1} + d_n}{10} = \frac{\frac{d_1 d_2 d_3 \ldots d_n + 10^n d_n - d_n}{10^n - 1}}{10}$$

$$= \frac{\frac{d_n d_1 d_2 \ldots d_n - d_n}{10^n - 1}}{10} = \frac{d_n d_1 d_2 \ldots d_{n-1}}{10^n - 1}$$

$$= 0.\overline{d_1 d_2 d_3 \ldots d_n} = r(x)$$

2b. By 2a, $r\left(\frac{1}{m}\right) = \frac{\frac{1}{m} + d_n}{10}$, where d_n is the last digit in the repetend of $\frac{1}{m}$. Since $\frac{\frac{1}{m} + d_n}{10} = \frac{1 + d_n}{10} \cdot \frac{1}{m}$, all we must show is that $1 + d_n * m$ is a multiple of 10. Since $\frac{1}{m} = 0.\overline{d_1 d_2 d_3 \ldots d_n} = \frac{d_1 d_2 d_3 \ldots d_n}{10^n - 1}$

$\Longrightarrow 10^n - 1 = m * (d_1 d_2 d_3 \ldots d_n) \Longrightarrow 10^n - m * (d_1 d_2 d_3 \ldots d_{n-1} 0) = m * d_n + 1.$

Since the left hand side of this equation is a multiple of 10, the right hand side must also be a multiple of 10. $r\left(\frac{1}{m}\right) = \frac{1 + d_n \cdot m}{10} \cdot \frac{1}{m}$ and $\frac{1 + d_n \cdot m}{10}$ is an integer.

2c. By 2b, $r\left(\frac{s}{m}\right) = \frac{s/m \cdot d_n}{10} = \frac{s + d_n \cdot m}{10} \cdot \frac{1}{m}$. Again $s + d_n * m$ is a multiple of 10 because

$\frac{s}{m} = \frac{d_1 d_2 d_3 \ldots d_n}{10^n - 1} \Rightarrow s \cdot 10^n - s = m \cdot (d_1 d_2 d_3 \ldots d_n) \Rightarrow s \cdot 10^n - m \cdot (d_1 d_2 d_3 \ldots d_{n-1} 0) = m \cdot d_n + s$. Since the left hand side of this equation is a multiple of 10, the right hand side must also be a multiple of 10. Thus the formula for t is $\frac{s + d_n \cdot m}{10}$ and t is guaranteed to be an integer.

3a. $153846 * 4 = 615384$ and $102564 * 4 = 410256$ (See 3b for general method.)

3b. From 3a, both $0.\overline{153846}$ and $0.\overline{102564}$ must be solutions. To help getting these into lowest terms, note that $r(x) = 4x \Longrightarrow \frac{x + d}{10} = 4x \Longrightarrow x = d/39$. Knowing this, it is easy to calculate that the first solution is 2/13 and the second is 4/39.

4a. We show if $s \equiv 1 \bmod 9$, then r (s / 2187) is some u / 2187 with the numerator $u \equiv 1 \bmod 9$. This suffices to prove the problem since $1 \equiv 1 \ mod 9$ and so all successive rotations of 1/2187 will have numerators congruent to 1 mod 9. Suppose $s \equiv 1 \ mod 9$. Then $r\left(\frac{s}{2187}\right) = \frac{s + 2187d}{10} \cdot \frac{1}{2187}$. We must show $\frac{s+2187}{10} \equiv 1 \ mod 9$.

Since $s \equiv 1 \ mod 9$ and $2187d \equiv 0 \ mod 9$, $s + 2187d \equiv 1 \ mod 9$. Since 10 and 9 are relatively prime, (s + 2187) / 10 = 1 mod 9. So r (s / 2187) has a numerator that is congruent to 1 mod 9.

4b. From the fact stated with the problem, there are 243 distinct rotations of 1/2187. Since there are 243 integers of the form $1 + 9k$ in the set {1, 2, 3, ..., 2186} and 4a states that only such integers may appear as numerators of the rotations of 1/2187, by the pigeon-hole principle, the rotations must be exactly the set {$(9k + 1)$ / 2187 with k = 0, 1, 2, ..., 242}.

4c. By 4b, the elements in the set of rotations of 1/2187 are spaced 1/243 apart. Since 1/243 < 1/100, there must be at least one element of the set in any interval of length 1/100. So some rotation of 1/2187 must fall in the interval [$0.d_1d_2$, $0.d_1d_2$ + 1/100). So for each digit pair d_1d_2, there is a rotation of 1/2187 which begins with $0.d_1d_2...$. So d_1d_2 must appear somewhere in the expansion of 1/2187.

<u>Supplementary Notes</u>

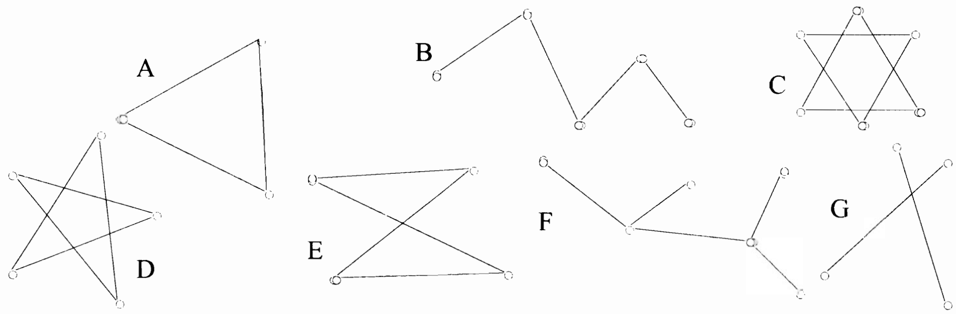
I. In this problem we dealt with only those fractions that have totally repeating decimal expansions. It is an interesting question to ask which fractions have this property. If $\frac{a}{b}$ is in lowest terms and $\frac{a}{b} = 0.\overline{d_1d_2 \ldots d_n}$, then $\frac{a}{b} = \frac{d_1d_2 \ldots d_n}{10^n - 1}$ and $b | 10^n - 1$, which implies every divisor of b also divides $10^n - 1$. Since $2 | 10^n$ and $5 | 10^n$, we know that 2 does not divide $10^n - 1$ and 5 does not divide $10^n - 1$. So b cannot be 1 or a multiple of 2 or 5 if $\frac{a}{b}$ is to have a totally repeating decimal. But are these conditions enough to guarantee that $\frac{a}{b}$ will be totally repeating? See any university number theory textbook (e.g. The Higher Arithmetic by Davenport) for the complete discussion and proof. Other interesting facts you might want to try to prove is that $\frac{a}{b}$ has a terminating decimal expansion if b is divisible by no primes other than 2 or 5 and $\frac{a}{b}$ is a partially repeating decimal if b is divisible by 2 or 5 and some other prime factor not equal to 2 or 5.

II. The difficult fact you were allowed to use in problem 4 holds more generally: $\frac{1}{3^t}$ has a repetend of length 3^{t-2} for any $t \geq 2$. As examples, $\frac{1}{9} = .\overline{1}$, $\frac{1}{27} = .\overline{037}$, and $\frac{1}{81} = .\overline{012345679}$. Equivalently, the smallest integer n for which $3^t | 10^n - 1$ is 3^{t-2}. Restating once more, the smallest integer n for which $10^n \equiv 1 \ mod \ 3^t$ is $n = 3^{t-2}$. This generalization allows us to prove many statements similar to problem 4c. Since $\frac{1}{10^3} > \frac{1}{3^8}$, the decimal expansion of $\frac{1}{3^{10}}$ contains every 3 digit sequence and since $\frac{1}{10^{100}} > \frac{1}{3^{498}}$ the decimal expansion of $\frac{1}{3^{500}}$ contains every 100-digit sequence.

Regular Closed Linkages

The Definitions and Theorems

A linkage is a figure made up of line segments connected at their endpoints. The figures A, B, D, E, and F below are examples of linkages. Figures C and G are not linkages but each consists of two overlayed linkages.



A linkage is closed if each endpoint is shared by exactly two segments. Figures A, D, and E are closed linkages. Figures B and F are not closed linkages.

A closed linkage is regular if all segments in the linkage are crossed by other segments the same number of times. In the figures above, only Figures A and D are regular closed linkages (RCL).

The symbol $[n, k]$, where n and k are non-negative integers, can be used to represent a (RCL) made up of n segments where each is crossed by other segments k times. Figure A is a [3, 0] RCL and Figure D is a [5, 2] RCL.

In this set of problems, we will investigate for which values of n and k is it possible to construct a $[n,, k]$ RCL.

The following theorems have evolved in the study of RCL's and may aid in your investigation and used in proving other theorems:

Theorem 1 If a $[n_1, k]$ RCL and a $[n_2, k]$ RCL are constructible, then a $[n_1 + n_2, k]$ RCL is also constructible.

Theorem 2 If for some integer m, a $[2m + 1, k]$ RCL is constructible, then a $[4m + 2, 2k + 1]$ RCL is also constructible.

ARML Power Contest – January 1997 – Regular Closed Linkages

The Problems:

1a. Construct a [7, 2] RCL.

1b. Construct a [7, 4] RCL.

2a. Construct a [6, 1] RCL.

2b. Construct a [8, 4] RCL.

3a. Construct a [10, 1] RCL.

3b. Construct a [12, 1] RCL.

4a. Prove it is impossible to construct a [5, 1] RCL.

4b. Prove it is impossible to construct a [6, 2] RCL. (4b may be the most difficult problem of the set.)

5. Prove it is always possible to construct a [$2n$, 1] RCL for any integer, n, greater than or equal to 5.

6a. Construct a [8, 2] RCL.

6b. Prove it is always possible to construct a [n, 2] RCL for any integer, n, greater than or equal to 7.

7a. Prove Theorem 1.

Given two RCL's, [n_1, k] and [n_2, k], show how to construct a [$n_1 + n_2$, k] RCL.

7b. Prove Theorem 2.

Given a [$2m + 1$, k] RCL, where m is a positive integer, show how to construct a [$4m + 2$, $2k + 1$] RCL.

The Extensions (not part of the contest problem)

1. Does there exist a positive integer n for which a [n, 3] RCL can be constructed?

2a. Construct a [8, 1] RCL.

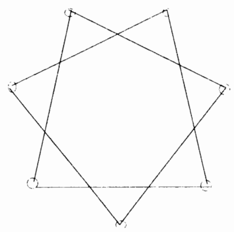
2b. If a [$2m + 1$, 0] RCL is constructible, then a [$4(m+1)$,1] RCL is also constructible.

3. If a [n, jk] RCL is constructible, then a [jn, k] RCL is also constructible for positive integers n, j, k.

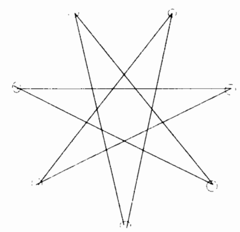
4. If $\gcd(m, n) = 1$ and $n \geq 2m + 1$, then a [n, $2(m - 1)$] RCL is always constructible.

The Solutions

1a.



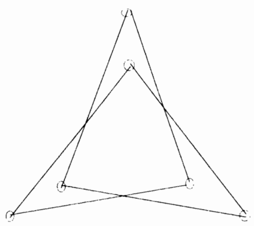
1b.



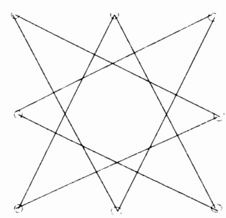
In general, $[2m + 1, 2]$ is constructible for $m \geq 2$.

In general, $[2m + 3, 2m]$ is constructible for $m \geq 1$.

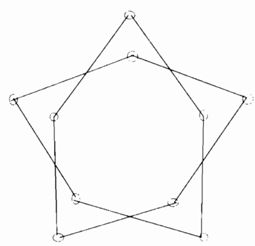
2a.



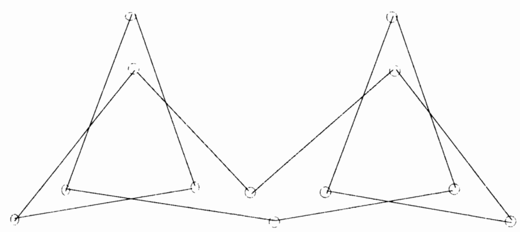
2b.



3a.



3b.

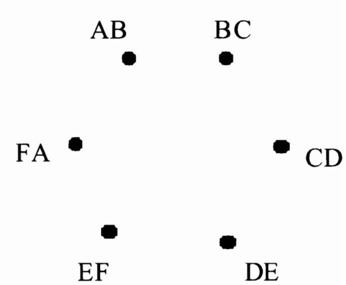


4a. In a $[n, k]$ RCL each of the n segments is crossed k times. Therefore, there must be n times k divided by 2 crossing points (because each point is counted twice). Since the number of crossing points must be a whole number, either n or k must be even. Therefore, [5, 1] is impossible.

4b. Consider a graph called the dual of [6, 2]. In this graph each segment of [6, 2] , AB, BC, CD, DE, EF, and FA, is represented by a point on a circle. Two points in this graph are connected if the segments they represent in [6, 2] cross. Since in [6, 2] each segment gets crossed twice, each point in the dual of [6, 2] must be connected to two (and only two) other points.

Lemma There are only two possible duals of [6, 2] (disregarding rotations of the labels).

Proof: Since adjacent points in the dual represent segments with common endpoints in [6, 2], they cannot be connected in the dual of [6, 2]. Therefore, AB must be connected with CD and DE or AB must be connected with CD and EF. (Connecting AB to DE and EF would be equivalent to AB to CD and DE and therefore does not need to be considered.)



Case 1 AB is connected to CD and DE.

Subcase 1a CD is connected to EF. Then EF must be connected to BC and then BC would have to be connected to FA and FA to DE.

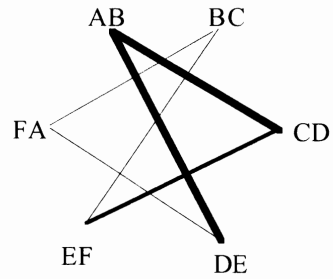


Figure 1a

Subcase 1b CD is connected to FA. Then FA must be connected to BC and then BC would have to connected to EF. This would force EF to be connected to DE, a contradiction.

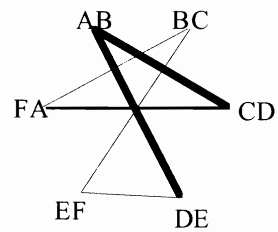


Figure 1b

Case 2 AB is connected to CD and EF
Then EF must be connected to CD or BC.

Subcase 2a EF is connected to CD. Then BC, DE and FA must be connected forming two overlapping triangles.

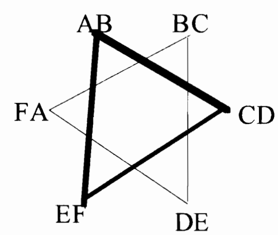


Figure 2a

Subcase 2b EF is connected to BC. Then BC can be connected to FA or DE.

Subsubcase 2ba BC is connected to FA. Then FA must be connected to DE. This would force DE to be connected to CD, a contradiction.

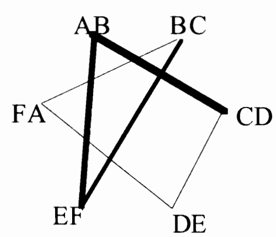


Figure 2ba

Subsubcase 2bb BC is connected to DE. Then DE must be connected to FA and FA to CD.

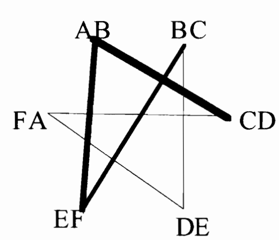


Figure 2bb

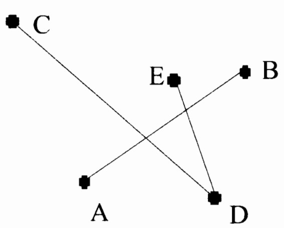
Notice that the graph in Subcase 2bb is just a rotation of the labels of the graph in Subcase 1a. Therefore, they can be considered the same graph.

Therefore, [6, 2] has only two possible duals as shown in Figures 1a and 2a.

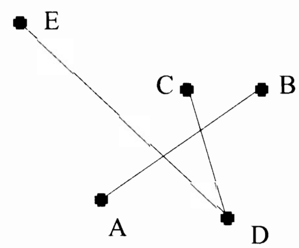
<u>Theorem</u> It is impossible to construct a [6, 2] RCL.

The duals found in the lemma show that AB must be crossed by CD and DE or AB must be crossed by CD and EF.

<u>Case 1</u> AB is crossed by CD and DE. This can be done in two possible ways:

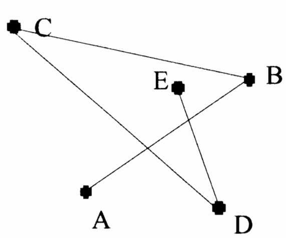


Subcase 1a

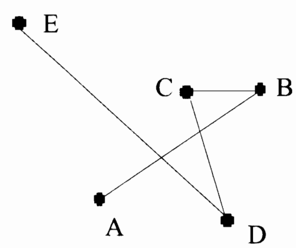


Subcase1b

Now draw in BC in each of the two possible cases. (It cannot cross DE since according to the dual BC crosses only EF and FA.)



Subcase 1a

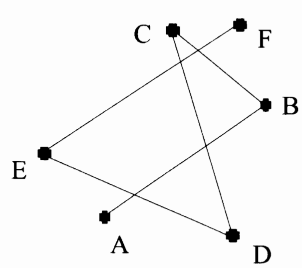


Subcase 1b

According to the dual EF must intersect CD and BC.

In subcase 1a, this is impossible because E is in the interior of Angle BCD and EF would have to intersect both sides of this angle. This happens only at vertex C.

However, in subcase1b it is possible to draw EF intersecting both CD and BC if points C and E are moved in the diagram.



Finally, according to the dual, AF must cross DE and BC. This is also possible, but in doing so AF must also cross CD. Would it be possible move points (and the accompanying segments) so this would not happen? The answer this time is no, because in the original diagram of this case, A and E are on one side of CD and F must be on the other side of CD since EF and CD cross.

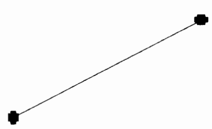
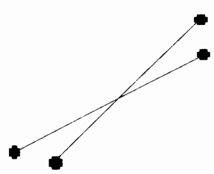
<u>Case 2</u> AB is crossed by CD and EF. Then either D and E are on the same side of AB or opposite sides of AB.

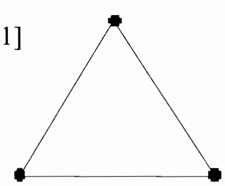
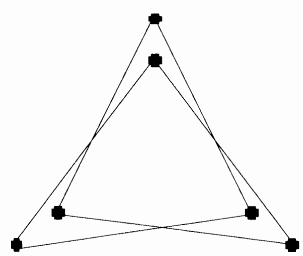
<u>Subcase 2a</u> D and E are on the same side of AB. Then C and F are on the other side of AB. Segment DE will be on one side of AB while FA and BC would be on the other and hence could not cross DE. A contradiction.

<u>Subcase 2b</u> D and E are on opposite sides of AB. Then forming DE (without crossing AB) forces all the other points to be on the same side of DE and the segments connecting these points could not cross DE. Again a contradiction.

Therefore, it is impossible to construct [6, 2].

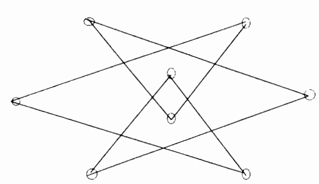
5. If k is odd ($k \geq 3$), it is always possible to construct a [k, 0] RCL as a convex polygon with k sides. By replacing each of the sides with a pair of crossing segments, as shown in the diagram, a [$2k$,1] will be constructed.

Replace  with 

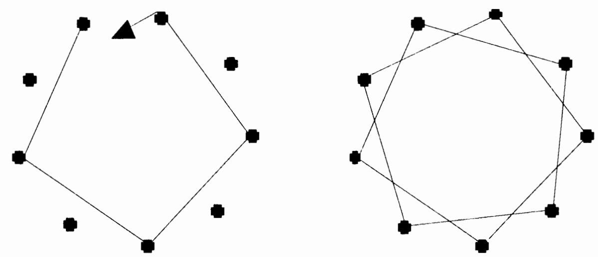
For example, [3, 0] to [6, 1]  

This still leaves the [$2k$, 1] RCLs where k is even. Each of these can be constructed by combining two previously constructed [$2k$, 1] RCLs using theorem 1.

6a.



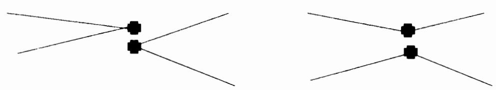
6b. If n is odd, place the n points around a circle and connect every other point. After the second trip around the circle you will have come back to the starting point and each segment will have two crossings. This works for any odd $n \geq 5$.



For example, [9, 2]:

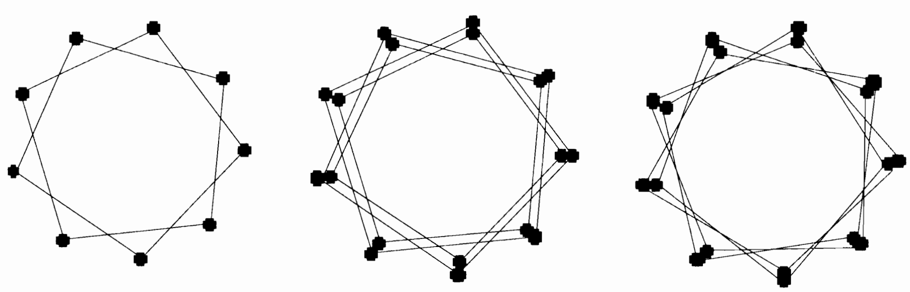
If n is even, [8,2] is a special case and shown in part 6a. If n is even and greater than 8, it can always be expressed as a sum of two odd integers, $n1$ and $n2$, greater than or equal to 5. The construction above shows that [$n1$, 2] and [$n2$, 2] are constructible and theorem 1 states that [$n1 + n2$, 2] will always be constructible when [$n1$, 2] and [$n2$, 2] are constructible.

7a. Place [$n1$, k] and [$n2$, k] next to each other (not overlapping) so that one endpoint from each of the RCLs is close to the other, one above the other. Move two of the four segments connected to these endpoints so that the two upper segments go to the upper endpoint and the two lower segments go to the lower endpoint. (See the diagram.)



Since no new points have been added and no crossings have been formed or deleted, the figure would be [$n1 + n2$, k].

7b. Make a copy of the [$2m + 1$, k] RCL but make it a little larger. Overlay these two copies, lining up the parallel sides. There are now $4m + 2$ endpoints and each segment should now have $2k$ crossings. Replace each set of parallel segments with two intersecting segments, producing the [$4m + 2$, $2k + 1$] RCL. (See the diagram.)

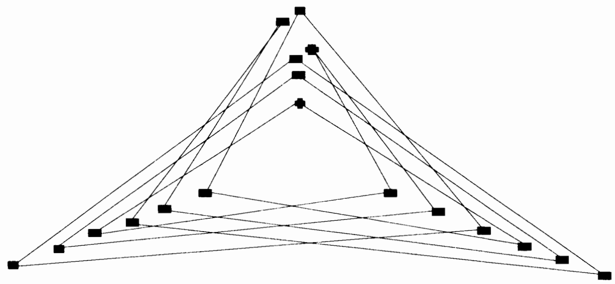


Because n in the original RCL is odd, you are guaranteed to get a RCL using this method. If n had been even, using this method would only produce two overlapping [n, k] RCLs.

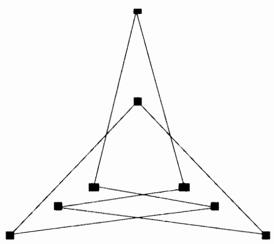
ARML Power Contest – January 1997 – Regular Closed Linkages

The Extensions

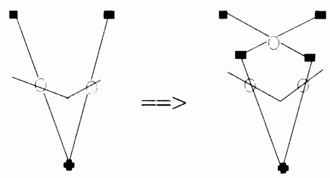
1. A general method has been found to generate $[n, 3]$ RCLs. The following [16, 3] RCL should serve as a spring-board for you investigations: It started with a [6, 1] RCL and traced twice, going to the outside after almost closing the linkage each time.



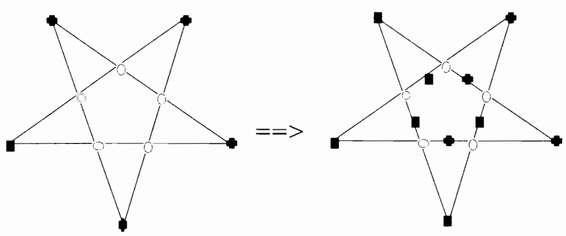
2a.



2b. From theorem 2, if $[2m + 1, 0]$ is constructible, then $[4m + 2, 1]$ is constructible. You can always add two more intersecting segments as indicated in the diagram:



3. The following diagram should serve as a springboard to further investigation. Changing [5, 2] to [10, 1]:



4. Put the n points around a circle and connect every m-th point. This provides $(2m - 1)$ crossings and is closed when $\gcd(m, n) = 1$.

Special thanks to Grisha Chelnokov from the Moscow Youth Science Center for the original idea behind the problem and to Davide Cervone from the University of Minnesota Geometry Center for his assistance and encouragement in the writing of this problem.

Factorial Polynomials

The Definitions

Factorial polynomials are defined as follows:

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = x(x-1)$

$P_3(x) = x(x-1)(x-2)$, etc.

In general, $P_n(x) = x(x-1)(x-2)...(x-n+1)$, and recursively, $P_n(x) = P_{n-1}(x)(x-n+1)$ and $P_0(x) = 1$.

Factorial polynomials can also be written in expanded form:

$P_0(x) = 1$

$P_1(x) = 1x + 0$

$P_2(x) = 1x^2 - 1x + 0$

$P_3(x) = 1x^3 - 3x^2 + 2x + 0$

$P_4(x) = 1x^4 - 6x^3 + 11x^2 - 6x + 0$

$P_5(x) = 1x^5 - 10x^4 + 35x^3 - 50x^2 + 24x + 0$,

creating the following triangle of coefficients:

1					
1	0				
1	-1	0			
1	-3	2	0		
1	-6	11	-6	0	
1	-10	35	-50	24	0 ...

Let $\left\{ n \atop k \right\}$ = the coefficient of x^{n-k} in the expansion of $P_n(x)$, with $0 \leq k \leq n$.

For example, $\left\{ 4 \atop 2 \right\} = 11$, $\left\{ 5 \atop 3 \right\} = -50$, and $\left\{ n \atop 0 \right\} = 1$.

The Problems

1a. Show that $P_3(x+1) - P_3(x) = 3P_2(x)$.

1b. Show that $P_4(x+1) - P_4(x) = 4P_3(x)$.

1c. Generalize the above two statements and prove your generalization is true.

2a. Show that $P_2(1) + P_2(2) + P_2(3) + \ldots + P_2(k) = \dfrac{P_3(k+1)}{3}$.

2b. Show that $P_3(1) + P_3(2) + P_3(3) + \ldots + P_3(k) = \dfrac{P_4(k+1)}{4}$.

2c. Generalize the above two statements and prove your generalization is true.

3a i. Show that $P_1(x) + P_2(x) = x^2$.

3a ii. <u>Use the above statement</u> to derive a simple, explicit formula for the sum $1^2 + 2^2 + 3^2 + \ldots + k^2$ in terms of k.

3b i. Show that $P_1(x) + 3P_2(x) + P_3(x) = x^3$.

3b ii. <u>Use the above statement</u> to derive a simple, explicit formula for the sum $1^3 + 2^3 + 3^3 + \ldots + k^3$ in terms of k.

3c. Express x^4 as the sum of factorial polynomials.

4a i. For $n \geq 1$, $\left\{ n \atop n \right\} =$ ________

4a ii. Explain this answer in terms of the roots of $P_n(x)$.

4b i. For $n \geq 2$, $\left\{ n \atop 1 \right\} =$ ________

4b ii. Explain this answer in terms of the roots of $P_n(x)$.

4c i. For $n \geq 2$, $\left\{ n \atop n-1 \right\} =$ ________

4c ii. Explain this answer in terms of the roots of $P_n(x)$.

4d i. For $n \geq 2$, $\sum_{k=0}^{n} \left\{ n \atop k \right\} =$ ________

4d ii. Explain this answer in terms of the roots of $P_n(x)$.

5. In Pascal's Triangle, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Likewise in this triangle, $\left\{ n \atop k \right\}$ can be derived from $\left\{ n-1 \atop k-1 \right\}$ and $\left\{ n-1 \atop k \right\}$. Complete this statement: For $n \geq 2$, $\left\{ n \atop k \right\} =$ ______ $\left\{ n-1 \atop k-1 \right\} +$ ______ $\left\{ n-1 \atop k \right\}$.

The Solutions

1a. $P_3(x+1) - P_3(x) = (x+1)(x)(x-1) - (x)(x-1)(x-2)$

$= (x)(x-1)((x+1) - (x-2))$

$= (x)(x-1)(3)$

$= 3P_2(x)$

1b. $P_4(x+1) - P_4(x) = (x+1)(x)(x-1)(x-2) - (x)(x-1)(x-2)(x-3)$

$= (x)(x-1)(x-2)((x+1) - (x-3))$

$= (x)(x-1)(x-2)(4)$

$= 4P_3(x)$

1c. $P_n(x+1) - P_n(x) = (x+1)(x)(x-1)\ldots(x+1-n+1) - (x)(x-1)(x-2)\ldots(x-n+1)$

$= (x)(x-1)\ldots(x+1-n+1)((x+1) - (x-n+1))$

$= (x)(x-1)\ldots(x-n+2)(x+1-x+n-1))$

$= (x)(x-1)\ldots(x-n+2)(n)$

$nP_{n-1}(x) = n(x)(x-1)\ldots(x-(n-1)+1)$

$= n(x)(x-1)\ldots(x-n+2)$

$\therefore P_n(x+1) - P_n(x) = nP_{n-1}(x)$

2a. $P_2(1) + P_2(2) + P_2(3) + \ldots + P_2(k) = \dfrac{P_3(k+1)}{3}$

$1(1-1) + 2(2-1) + 3(3-1) + \ldots + k(k-1) = \dfrac{(k+1)(k)(k-1)}{3}$

Prove : For all $k \geq 1$, $0 + 2 + 6 + 12 + \ldots + k(k-1) = \dfrac{(k+1)(k)(k-1)}{3}$.

1) Show true for $k = 1$.

$$0 = \frac{(1+1)(1)(1-1)}{3}$$

2) Assume $0 + 2 + 6 + 12 + \ldots + j(j-1) = \dfrac{(j+1)(j)(j-1)}{3}$, prove

$$0 + 2 + 6 + 12 + \ldots + j(j-1) + (j+1)(j) = \frac{(j+2)(j+1)(j)}{3}.$$

$$\frac{(j+1)(j)(j-1)}{3} + (j+1)(j) = \frac{(j+2)(j+1)(j)}{3}$$

$$\frac{(j+1)(j)(j-1)}{3} + \frac{3(j+1)(j)}{3} = \frac{(j+2)(j+1)(j)}{3}$$

$$\frac{(j+1)(j)(j-1+3)}{3} = \frac{(j+2)(j+1)(j)}{3}$$

$$\frac{(j+1)(j)(j+2)}{3} = \frac{(j+2)(j+1)(j)}{3}$$

2b. $P_3(1) + P_3(2) + P_3(3) + \ldots + P_3(k) = \dfrac{P_4(k+1)}{4}$

$$1(0)(-1) + 2(1)(0) + 3(2)(1) + \ldots + k(k-1)(k-2) = \frac{(k+1)(k)(k-1)(k-2)}{4}$$

Prove: $0 + 0 + 6 + 24 + \ldots + k(k-1)(k-2) = \dfrac{(k+1)(k)(k-1)(k-2)}{4}$.

1) Show true for $k = 1$.

$$0 = \frac{(2)(1)(0)(-1)}{4}$$

2) Assume $0 + 0 + 6 + 24 + \ldots + j(j-1)(j-2) = \dfrac{(j+1)(j)(j-1)(j-2)}{4}$, prove

$$0 + 0 + 6 + 24 + \ldots + (j+1)(j)(j-1) = \frac{(j+2)(j+1)(j)(j-1)}{4}.$$

$$\frac{(j+1)(j)(j-1)(j-2)}{4} + (j+1)(j)(j-1) = \frac{(j+2)(j+1)(j)(j-1)}{4}$$

$$\frac{(j+1)(j)(j-1)(j-2)}{4} + \frac{4(j+1)(j)(j-1)}{4} = \frac{(j+2)(j+1)(j)(j-1)}{4}$$

$$\frac{(j+1)(j)(j-1)((j-2)+4)}{4} = \frac{(j+2)(j+1)(j)(j-1)}{4}$$

$$\frac{(j+1)(j)(j-1)((j+2)}{4} = \frac{(j+2)(j+1)(j)(j-1)}{4}$$

2c. Prove for a given positive n, $P_n(1) + P_n(2) + P_n(3) + \ldots + P_n(k) = \dfrac{P_{n+1}(k+1)}{n+1}$ for any k.

a) Show true for $k = 1$.

Show for any n , $P_n(1) = \dfrac{P_{n+1}(2)}{n+1}$

If $n = 0, P_0(1) = 1$ and $\dfrac{P_1(2)}{0+1} = 2$ (not true when $n = 0$!)

If $n = 1, P_1(1) = 1$ and $\dfrac{P_2(2)}{1+1} = 1$

If $n \geq 2, P_n(1) = 0$ and $P_{n+1}(2) = 0$

b) Assume $P_n(1) + P_n(2) + P_n(3) + \ldots + P_n(j) = \dfrac{P_{n+1}(j+1)}{n+1}$, prove

$$P_n(1) + P_n(2) + P_n(3) + \ldots + P_n(j) + P_n(j+1) = \frac{P_{n+1}(j+2)}{n+1}.$$

$$\frac{P_{n+1}(j+1)}{n+1} + P_n(j+1) = \frac{P_{n+1}(j+2)}{n+1}$$

$$(n+1)P_n(j+1) = P_{n+1}(j+2) - P_{n+1}(j+1)$$
$$= (j+2)(j+1)(j)(j-1)\ldots(j+2-(n+1)+1) - (j+1)(j)(j-1)\ldots(j+1-(n+1)+1)$$
$$= (j+2)(j+1)(j)(j-1)\ldots(j-n+2) - (j+1)(j)(j-1)\ldots(j-n+1)$$
$$= (j+1)(j)(j-1)\ldots(j-n+2)(j+2-(j-n+1))$$
$$= (j+1)(j)(j-1)\ldots(j-n+2)(n+1)$$
$$= P_n(j+1)(n+1)$$

3a i. $P_1(x) + P_2(x) = x + x(x-1)$

$$= x + x^2 - x$$
$$= x^2$$

3a ii. $1^2 + 2^2 + 3^2 + \ldots + k^2 =$

$$P_1(1) + P_2(1) + P_1(2) + P_2(2) + P_1(3) + P_2(3) + \ldots + P_1(k) + P_2(k) =$$

$$P_1(1) + P_1(2) + P_1(3) + \ldots + P_1(k) + P_2(1) + P_2(2) + P_2(3) + \ldots + P_2(k) =$$

From problem 2 above,

$$\frac{P_2(k+1)}{2} + \frac{P_3(k+1)}{3}$$

$$\frac{(k+1)(k)}{2} + \frac{(k+1)(k)(k-1)}{3}$$

$$\frac{3(k+1)(k) + 2(k+1)(k)(k-1)}{6} = \frac{k(k+1)(2k+1)}{6}$$

3b i. $P_1(x) + 3P_2(x) + P_3(x) = (x) + 3(x)(x-1) + x(x-1)(x-2)$

$$= x(1 + 3x - 3 + x^2 - 3x + 2)$$
$$= x(x^2) = x^3$$

3b ii. $1^3 + 2^3 + \ldots + k^3 =$

$$P_1(1) + 3P_2(1) + P_3(1) + P_1(2) + 3P_2(2) + P_3(2) + \ldots + P_1(k) + 3P_2(k) + P_3(k) =$$

$$P_1(1) + P_1(2) + \ldots + P_1(k) + 3P_2(1) + 3P_2(2) + \ldots + 3P_2(k) + P_3(1) + P_3(2) + \ldots + P_3(k) =$$

$$\frac{P_2(k+1)}{2} + 3\left(\frac{P_3(k+1)}{3}\right) + \frac{P_4(k+1)}{4} =$$

$$\frac{2(k+1)(k) + 4(k+1)(k)(k-1) + (k+1)(k)(k-1)(k-2)}{4} = \frac{k^4 + 2k^3 + k^2}{4} = \left(\frac{k(k+1)}{2}\right)^2$$

3c. $x^4 = P_1(x) + 7P_2(x) + 6P_3(x) + P_4(x)$

$= P_1(x) + aP_2(x) + bP_3(x) + P_4(x)$

$= x + ax(x-1) + b(x)(x-1)(x-2) + x(x-1)(x-2)(x-3)$

$= x(1 + ax - a + bx^2 - 3bx + 2b + x^3 - 6x^2 + 11x - 6) \ldots$

If $(1 + ax - a + bx^2 - 3bx + 2b + x^3 - 6x^2 + 11x - 6)$ is to equal x^3, then

$1 - a + 2b - 6 = 0$, $a - 3b + 11 = 0$, and $b - 6 = 0$. Therefore, $a = 7$ and $b = 6$.

4a) $\left\{ {n \atop n} \right\} = 0$ In any polynomial the product of the roots is equal to the constant term divided by the leading coefficient. Since zero is a root of any factorial polynomial (when $n \geq 1$), this product will always be zero.

b) $\left\{ {n \atop 1} \right\} = -\frac{(n)(n-1)}{2} = T_{n-1}$, the n th Triangle Number. In any polynomial, the sum of the roots is equal to $-b/a$, where a is the leading coefficient and b is the coefficient of x^{n-1}. In any factorial polynomial, the roots are 0, 1, 2, 3, ..., $n - 1$ and the sum of these numbers is the $(n-1)$th Triangle Number.

c) $\left\{ {n \atop n-1} \right\} = (-1)^{n-1}(n-1)!$ In any polynomial the sum of all the products of the roots where one of the roots is left out of each product is equal to $(-1)^{n-1}\frac{y}{a}$, where y is the coefficient of the linear term and a is the leading coefficient. Since zero is always a root of a factorial polynomial (when $n \geq 2$)., all the products will equal zero except the product that has zero left out. This product will be $1(2)(3)(4)\ldots(n-1)$ or $(n-1)!$.

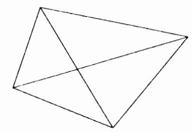
d) $\sum_{k=0}^{n} \left\{ {n \atop k} \right\} = 0$ In any polynomial if one is a root then the sum of all the coefficients will always be zero. Since one is a root of all factorial polynomials (where $n \geq 2$), the coefficients must add up to zero.

5. For $n \geq 2$, $\left\{ {n \atop k} \right\} = (1-n)\left\{ {n-1 \atop k-1} \right\} + (1)\left\{ {n-1 \atop k} \right\}$ (Found by inspection and trial and error!)

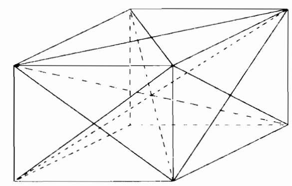
Integer Geometry

The Definitions and Theorems:

An integer polygon is a polygon whose sides (edges) and diagonals all have integer lengths.



An integer polyhedron is a polyhedron whose faces are all integer polygons and interior diagonals also have integer lengths.



The measure used to order such polygons and polyhedra is perimeter-plus which is the sum of the lengths of the edges added to the sum of the lengths of all the diagonals.

In the following problems the following theorems along with the Pythagorean Theorem and the Triangle Inequality may be useful:

Theorem 1: All isosceles trapezoids are cyclic, that is they can be inscribed in a circle.

Theorem 2: (Ptolemy's Theorem) In any cyclic quadrilateral the sum of the products of the opposite sides is equal to the product of the diagonals.

The Problems

1a. Find all integer rectangles whose perimeter-plus is ≤ 40. Prove there are no more. (Draw each rectangle, labeling the lengths of its sides and diagonals and next to each rectangle indicate its perimeter-plus.)

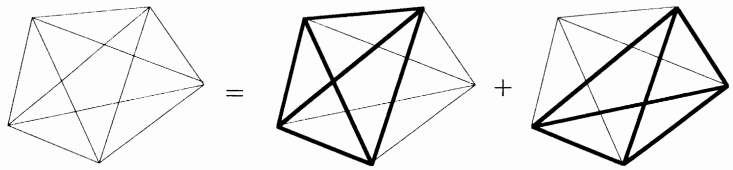
1b. Prove there are no integer squares.

1c. Find the smallest integer rhombus and calculate its perimeter-plus number. (Draw the rhombus, labeling the lengths of its sides and diagonals and next to it indicate its perimeter-plus.)

1d. Find all integer isosceles trapezoids whose perimeter-plus is ≤ 40. (Draw each trapezoid, labeling the lengths of its sides and diagonals and next to each trapezoid indicate its perimeter-plus.)

1e. There exist two small cyclic quadrilaterals with no parallel sides. Can you find them and their perimeter-plus numbers? (Hint: Both have diagonals 7 and 8 units long.) (Draw each quadrilateral, labeling the lengths of its sides and diagonals and next to each quadrilateral indicate its perimeter-plus.)

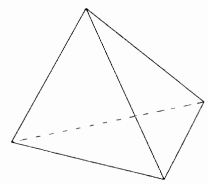
2. An integer pentagon can be formed by overlapping two integer quadrilaterals:



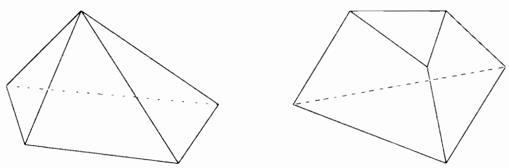
Using the quadrilaterals you found in problems 1d and 1e, find two integer pentagons and calculate their perimeter-plus number. (Draw each pentagon, labeling the lengths of its sides and diagonals and next to each pentagon indicate its perimeter-plus.)

3. Using a technique similar to the above, find the smallest integer hexagon. (Draw the hexagon, labeling the lengths of its sides and diagonals and next to it indicate its perimeter-plus.)

4. Since all tetrahedrons (3-D objects with four faces) have triangular faces and no interior diagonals, all tetrahedrons with integer length edges are integer tetrahedrons. A pentahedron (3-D objects with 5 faces) can be constructed in two ways:

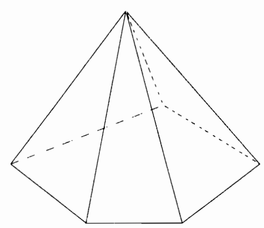


   i) a pyramid with four triangular sides and a quadrilateral base.
   ii) a frustum with three quadrilateral sides and two triangular bases.

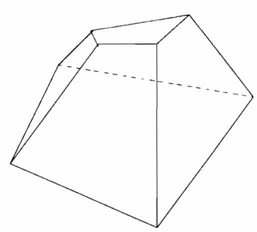


Find the smallest integer pentahedron of each type and calculate each figure's perimeter-plus number.

5a. A "pyramidal" hexahedron can be formed from a pentagonal base with five triangular sides. Find the smallest integer "pyramidal" hexahedron and calculate its perimeter-plus number.



5b. A "frustumal" hexahedron can be formed from six quadrilateral faces. Show how two 8 by 15 rectangles and four isosceles trapezoids with bases of 8 and 15 and sides of 7 can be arranged to form a "frustumal" hexahedron. Find its perimeter-plus number.



The Solutions

1a. There is only one...a 3 by 4 rectangle with a diagonal of 5.

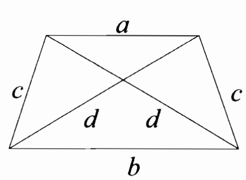
Proof: The sides and diagonal will form a right triangle and therefore must be a Pythagorean triple. All such triples can be written as $(2mn, m^2 - n^2, m^2 + n^2)$ where m, n are integers with $m > n > 0$. Since the perimeter-plus must be ≤ 40, $2mn + m^2 - n^2 + m^2 + n^2$ must be ≤ 20, and therefore, $m(m + n)$ must be less than or equal to 10. The smallest value for (m, n) is (2, 1) which is a solution to the inequality and produces the triple (4, 3, 5). The next smallest value for (m, n) is (3, 2), which produces the triple (12, 5, 13), is not a solution to the inequality and all larger values for (m, n) also fail to solve the inequality. Hence, the 3 by 4 rectangle is the only solution.

1b. The diagonal of a square is always $\sqrt{2}$ times the length of the side and can never be an integer if the side is an integer.

1c. The smallest rhombus has sides of 5 and diagonals of 6 and 8. It is important to know that the diagonals of a rhombus are perpendicular and bisect each other to convince yourself that this is indeed the smallest integer rhombus.

1d. Because current literature has two definitions for a trapezoid, the answer is either 8 or 9. If your definition states a trapezoid is a quadrilateral with only one pair of parallel sides, the answer is 8; however, if your definition states a trapezoid is a quadrilateral with a pair of parallel sides, the answer is 9 because all rectangles would be considered isosceles trapezoids.

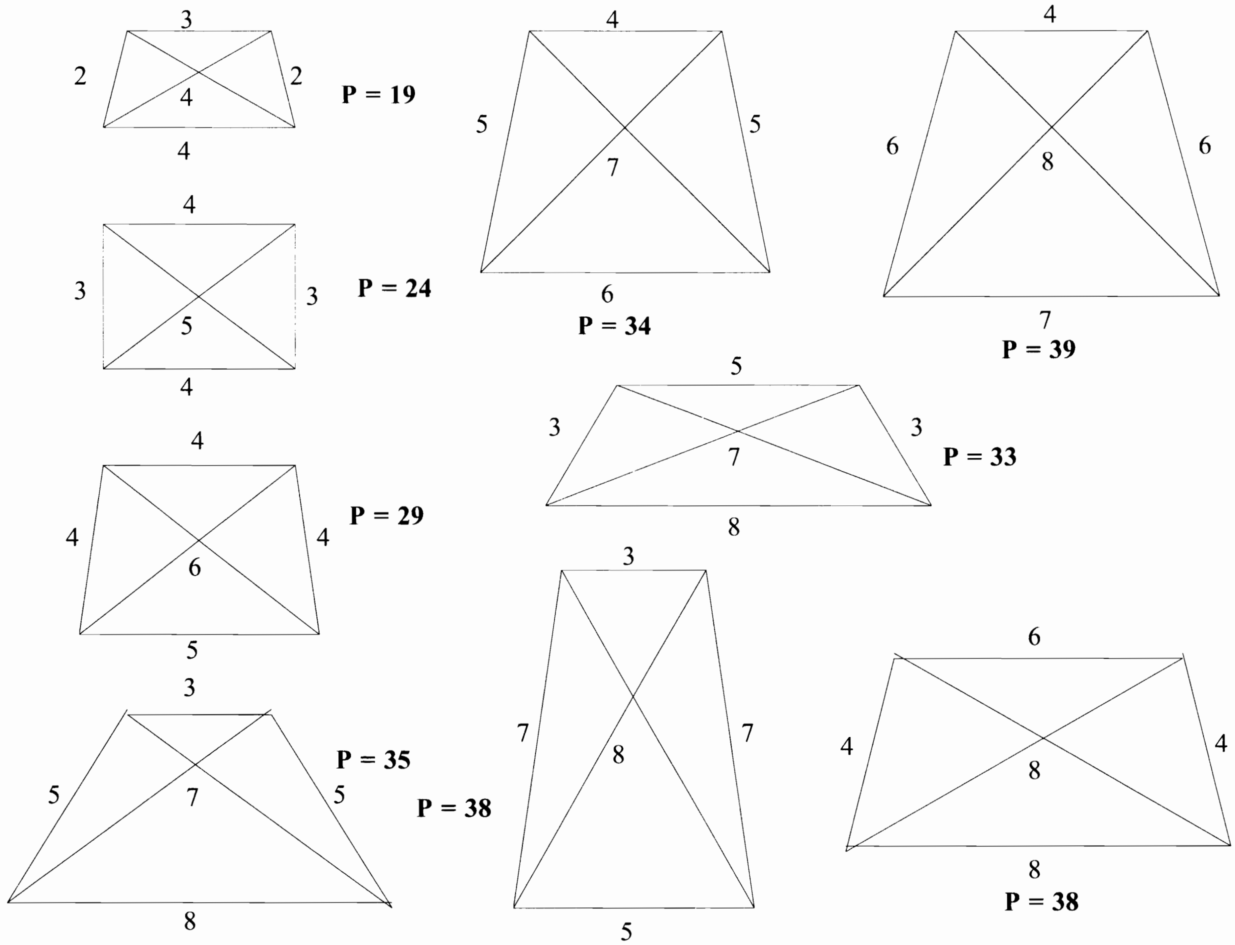
Since all isosceles trapezoids are cyclic, Ptolemy's Theorem holds. Their diagonals are also congruent. Therefore, using the labeling in the figure, $ab = d^2 - c^2$ and $a + c > d$. In order to count them all, consider the following table:



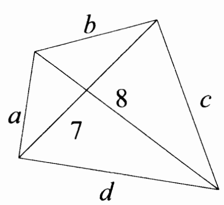
d	c	$(a)(b)$	a, b
3	2	5	1, 5
3	1	8	1, 8
			2, 4
4	3	7	1, 7
4	2	12	1, 12
			2, 6
			3, 4 *
5	4	9	3, 3 **
5	3	16	4, 4 **

d	c	$(a)(b)$	a, b
6	4	20	4, 5
7	5	24	3, 8
7	5	24	4, 6
7	3	40	5, 8
8	7	15	3, 5
8	6	28	4, 7
8	4	48	6, 8
9	6	45	5, 9
9	3	72	8, 9
10	6	64	8, 8
10	4	84	7, 12

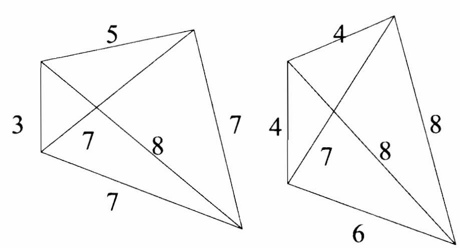
The first six entries in this table can be eliminated because $a + c \le d$. But the seventh entry(*) produces a solution where a = 3, b = 4, c = 2 , and d = 4. The next two solutions (**) are identical and produce the rectangle solution. The last four entries and all subsequent entries produce integer isosceles trapezoids but their perimeter-plus number is greater than 40. Here are the diagrams of these solutions:



1e. Using the labeled figure below, $ac + bd = 56$ and $a + b > 7$, $b + c > 8$, $a + d > 7$, and $c + d > 8$. Therefore, $c - a > 1$. This data produced this table :

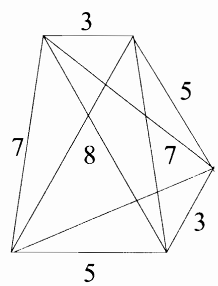


ac	+	bd	=	56
(3)(5)		(1)(41)		
(3)(6)		(2)(19)		
(3)(7)		(5)(7)*		
(3)(8)		(4)(8)		
(4)(6)		(4)(8)*		
(4)(7)		(4)(7)		

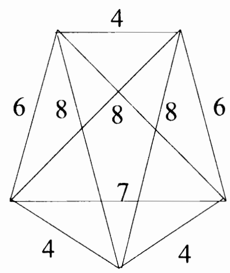


Of these six possibilities only the two starred entries satisfy all the inequalities (**P = 37** in both figures)

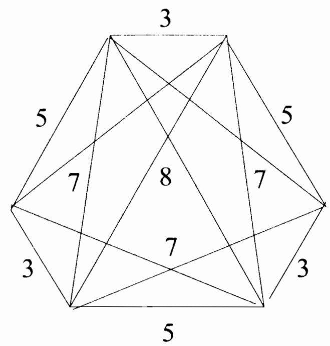
2.



P = 60



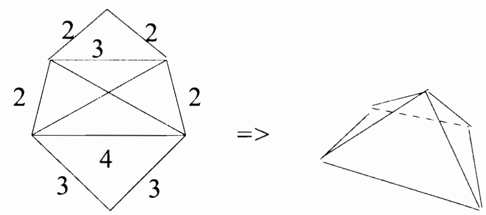
P = 63



P = 90

3. It is believed this hexagon was known to Euler. Note that it has the asymmetrical pentagon as a subset.

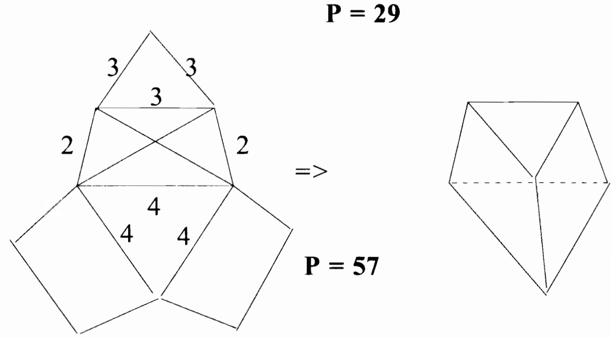
4a. Select the smallest integer quadrilateral for the base and place the smallest isosceles triangles on its parallel sides. The vertex of each of these triangles will become the vertex of the pyramid.



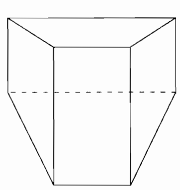
P = 29

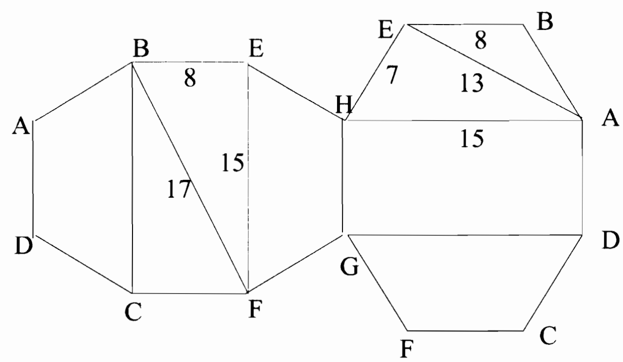
4b. Use the smallest integer quadrilaterals for the sides and equilateral triangles for the bases:

5a. Since the asymmetrical pentagon has the smallest perimeter-plus number, use it for the base of the pyramid and attach an isosceles triangle to each side. Since the pentagon has at least one diagonal of length 8, the congruent sides of all these isosceles triangles must be 5 to satisfy the Triangle Inequality. Therefore, **P = 85**



5b. Internal diagonals (like BG) are all diagonals on interior isosceles trapezoids (like BAGF) and will have length 17. Therefore the perimeter-plus number will be 4(8) + 4(15) + 4(7) + 4(17) + 4(13) + 4(17) = **P = 308**





For more information about integer geometry, see "Mathematical Gazette", vol 81, pp 18-28.

Unit Fractions

The Definition

A unit fraction is a fraction whose numerator is 1 and whose denominator is a positive integer. In other words, unit fractions are the reciprocals of the positive integers.

The Problems

1. As discovered in the Rhind Papyrus, dating from 1800 B.C., Egyptian mathematicians represented all positive fractions less than 1 as sums of distinct (different) unit fractions. For example, $\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$, $\frac{2}{3} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}$, and $\frac{5}{7} = \frac{1}{2} + \frac{1}{5} + \frac{1}{70}$. In 1202 A.D., Leonardo Fibonacci published a systematic way do this:

   If $0 < \frac{m}{n} < 1$, then $\frac{m}{n} = \frac{1}{q} + \{\text{representation of } \frac{m}{n} - \frac{1}{q}\}$, where $q = \lfloor \frac{n}{m} \rfloor$. This algorithm is repeated until the $\{\text{representation of } \frac{m}{n} - \frac{1}{q}\}$ is a unit faction.

Express each of the following as the sum of distinct unit fractions:

a) $\frac{7}{8}$ b) $\frac{5}{11}$ c) $\frac{7}{16}$

2. The unit fraction $\frac{1}{15}$ can be represented as the sum of two unit fractions in five different ways:

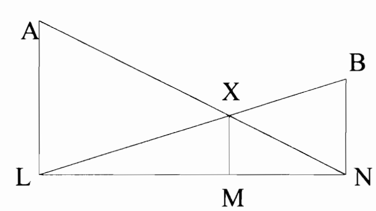
   $\frac{1}{15} = \frac{1}{30} + \frac{1}{30} = \frac{1}{20} + \frac{1}{60} = \frac{1}{24} + \frac{1}{40} = \frac{1}{18} + \frac{1}{90} = \frac{1}{16} + \frac{1}{240}$, but $\frac{1}{4}$ can be represented as the sum of two unit fractions in only three different ways: $\frac{1}{4} = \frac{1}{5} + \frac{1}{20} = \frac{1}{6} + \frac{1}{12} = \frac{1}{8} + \frac{1}{8}$.

a) Express $\frac{1}{12}$ as the sum of two unit fractions in eight ways ignoring the order of the addends.

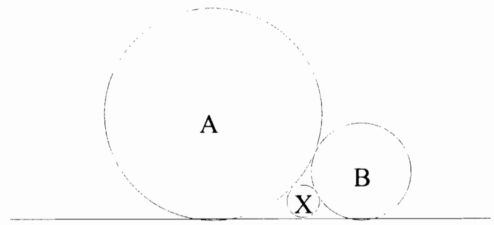
b) Express $\frac{1}{17}$ as the sum of two unit fractions in two ways ignoring the order of the addends.

c) In how many ways can $\frac{1}{n}$ be expressed as the sum of two unit fractions? Prove your assertion.

3. In the diagram at the right, $\overline{AL}$, $\overline{XM}$, and $\overline{BN}$ are parallel. If $\overline{AL} = a$, $\overline{BN} = b$, and $\overline{XM} = x$, prove $\frac{1}{x} = \frac{1}{a} + \frac{1}{b}$.



4. In the diagram at the right, each circle is tangent to the line and to the other two circles. If the radius of circle A is a, the radius of circle B is b, and the radius of circle X is x, prove $\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$.



5a. Prove: $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{k(k+1)} + \ldots = 1$.

(Hint: Use the fact that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ and rewrite each term.)

5b. Prove : $\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \ldots + \frac{2}{k(k+1)(k+2)} + \ldots = \frac{1}{2}$

(Hint: Again rewrite $\frac{2}{k(k+1)(k+2)}$ as the sum of two unit fractions.)

5c. Prove: In general, $\frac{j!}{k(k+1)(k+2)\ldots(k+j)} = \frac{(j-1)!}{k(k+1)(k+2)\ldots(k+j-1)} - \frac{(j-1)!}{(k+1)(k+2)\ldots(k+j)}$.

6. Consider the following triangle of unit fractions known as the Harmonic Triangle of Liebniz:

diagonals

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & \frac{1}{2} & & \frac{1}{2} & & & & \\
 & & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & & \\
 & & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & & \\
 & \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} & \\
\frac{1}{6} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{60} & & \frac{1}{30} & & \frac{1}{6}
\end{array}
$$

For any $n \geq k \geq 1$, $\left\{ n \atop k \right\}$ refers to the kth fraction in the nth row of this triangle, e.g. $\left\{ 5 \atop 3 \right\} = \frac{1}{30}$.

a) Generate the next two rows of the Harmonic Triangle.

b) Complete this equation: For any $n \geq k \geq 1$, $\left\{ n \atop k \right\} = \left\{ ? \atop ? \right\} + \left\{ ? \atop ? \right\}$.

c) $\left\{ n \atop 1 \right\} = \frac{1}{n}$, $\left\{ n \atop 2 \right\} = \frac{1}{n(n-1)}$, $\left\{ n \atop 3 \right\} = \frac{2}{n(n-1)(n-2)}$, etc. Find an explicit formula to calculate $\left\{ n \atop k \right\}$.

(Hint: Think factorials!)

d) Although there is no simple formula for the sum of the numbers in row n, $\sum_{k=1}^{n} \left\{ n \atop k \right\}$, there is a formula to determine the infinite sum of the numbers in any diagonal where k is held constant, $\sum_{n=k}^{\infty} \left\{ n \atop k \right\}$. (Note: The one exception is the first diagonal when $k = 1$. $\sum_{n=1}^{\infty} \left\{ {n \atop 1} \right\} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$, which is the harmonic series and is known to diverge, i.e., increases without any bound.)

i) For any $k \geq 2$, find a formula for $\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\}$.

ii) Prove the formula is true.

The Extensions (Not part of the contest)

The following are interesting facts about unit fractions taken from The Penguin Dictionary of Curious and Interesting Numbers by David Wells:

a) $\frac{2}{3}$ is the only positive fraction less than 1 that cannot be expressed as the sum of two unique unit fractions and therefore was a uniquely represented "Egyptian" fraction.

b) Erdos and Sierpinski have conjectured, respectively, that $\frac{4}{n}$ and $\frac{5}{n}$ are each expressible as the sum of three unit fractions.

c) $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$

d) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots$ (Leibniz - 1673)

e) $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ (Newton - 1665)

f) If n is an even number, $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} \ldots =$ a multiple of π^n. For example, if $n = 2$, the sum equals $\frac{\pi^2}{6}$ and if $n = 4$, the sum equals $\frac{\pi^4}{90}$. (Euler - 1736)

g) Every number greater than 77 is the sum of integers, the sum of whose reciprocals is 1. For example, $78 = 2 + 6 + 8 + 10 + 12 + 40$ and $\frac{1}{2} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{40} = 1$. (Graham - 1983)

h) A neat connection between Pascal's Triangle and the harmonic series:

$$(1 \times 1) = 1$$
$$(1 \times 1) - (1 \times \tfrac{1}{2}) = \tfrac{1}{2}$$
$$(1 \times 1) - (2 \times \tfrac{1}{2}) + (1 \times \tfrac{1}{3}) = \tfrac{1}{3}$$
$$(1 \times 1) - (3 \times \tfrac{1}{2}) + (3 \times \tfrac{1}{3}) - (1 \times \tfrac{1}{4}) = \tfrac{1}{4}$$
$$(1 \times 1) - (4 \times \tfrac{1}{2}) + (6 \times \tfrac{1}{3}) - (4 \times \tfrac{1}{4}) + (1 \times \tfrac{1}{5}) = \tfrac{1}{5}$$

The Solutions

1a. $\frac{7}{8}=\frac{1}{2}+\frac{3}{8}=\mathbf{\frac{1}{2}+\frac{1}{3}+\frac{1}{24}}$

1b. $\frac{5}{11}=\frac{1}{3}+\frac{4}{33}=\mathbf{\frac{1}{3}+\frac{1}{9}+\frac{1}{99}}$

1c. $\frac{7}{16}=\frac{1}{3}+\frac{5}{48}=\mathbf{\frac{1}{3}+\frac{1}{10}+\frac{1}{240}}$

2a. $\frac{1}{12}=\frac{1}{13}+\frac{1}{156}=\frac{1}{14}+\frac{1}{84}=\frac{1}{15}+\frac{1}{60}=\frac{1}{16}+\frac{1}{48}=\frac{1}{18}+\frac{1}{36}=\frac{1}{20}+\frac{1}{30}=\frac{1}{21}+\frac{1}{28}=\frac{1}{24}+\frac{1}{24}$

2b. $\frac{1}{17}=\frac{1}{34}+\frac{1}{34}=\frac{1}{18}+\frac{1}{306}$

2c. If $\frac{1}{n}=\frac{1}{x}+\frac{1}{y}$, then $x,y>n$. Let $x=n+j$ and $y=n+k$, then
$\frac{1}{n}=\frac{1}{n+j}+\frac{1}{n+k}\Rightarrow(n+j)(n+k)=n(n+k)+n(n+j)\Rightarrow jk=n^2$. Therefore, j and k are divisors of n^2. Since a square has an odd number of divisors, d, add 1 to this number (because j and k can both be n) and then divide by 2 since the order of j and k is insignificant. Therefore, if d is the number of divisors of n^2, then $\frac{1}{n}$ can be written as the sum of two unit fractions in $\mathbf{\frac{d+1}{2}}$ ways.

3. Let LM = m and MN = n, by similar triangles, $\frac{n}{x}=\frac{m+n}{a}$ and $\frac{m}{x}=\frac{m+n}{b}$ and so $m=\frac{an}{b}$. From the first equation, $x=\frac{an}{m+n}$. Therefore, $\frac{1}{x}=\frac{m+n}{an}$ and substituting for m, $\frac{1}{x}=\frac{\frac{an}{b}+n}{an}\Rightarrow\frac{1}{x}=\frac{an+bn}{abn}\Rightarrow\mathbf{\frac{1}{x}=\frac{1}{b}+\frac{1}{a}}$.

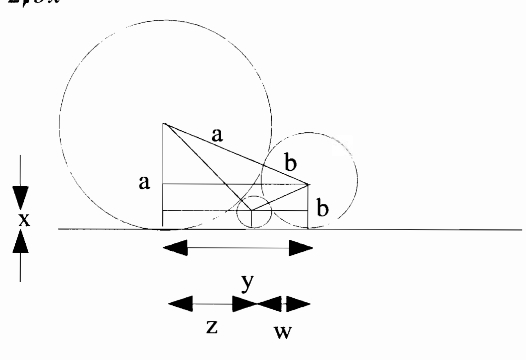
4. In the following diagram, using the Pythagorean Theorem three times, you get:

$(a-b)^2+y^2=(a+b)^2$, $y=2\sqrt{ab}$; $(a-x)^2+z^2=(a+x)^2$, $z=2\sqrt{ax}$; $(b-x)^2+w^2=(b+x)^2$, $w=2\sqrt{bx}$

But $y=z+w$

Therefore, $2\sqrt{ab}=2\sqrt{bx}+2\sqrt{ax}$

Dividing each term by $2\sqrt{abx}$, $\frac{1}{\sqrt{x}}=\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}$.



5a. For any k, $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$.

Using this fact, rewrite each term as the difference if two unit fractions:

$(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\ldots+(\frac{1}{k}-\frac{1}{k+1})+\ldots$

Summing the opposites, results in: $1-\frac{1}{k+1}$.

As $k\to\infty$, $\frac{1}{k+1}\to 0$. Therefore, the infinite sum is 1.

5b. For any k, it is easy to show : $\dfrac{2}{k(k+1)(k+2)} = \dfrac{1}{k(k+1)} - \dfrac{1}{(k+1)(k+2)}$

Using this fact, rewrite each term as the difference if two unit fractions:

$(\frac{1}{2} - \frac{1}{6}) + (\frac{1}{6} - \frac{1}{12}) + (\frac{1}{12} - \frac{1}{20}) + \ldots + (\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}) + \ldots$

Summing the opposites, results in: $\frac{1}{2} - \frac{1}{(k+1)(k+2)}$.

As $k \to \infty$, $\frac{1}{(k+1)(k+2)} \to 0$. Therefore, the infinite sum is $\frac{1}{2}$.

5c.
$$\frac{(j-1)!}{k(k+1)(k+2)\ldots(k+j-1)} - \frac{(j-1)!}{(k+1)(k+2)\ldots(k+j)} = \frac{(k+j)(j-1)! - k(j-1)!}{k(k+1)(k+2)\ldots(k+j)}$$
$$= \frac{k(j-1)! + j(j-1)! - k(j-1)!}{k(k+1)(k+2)\ldots(k+j)}$$
$$= \frac{j(j-1)!}{k(k+1)(k+2)\ldots(k+j)}$$
$$= \frac{j!}{k(k+1)(k+2)\ldots(k+j)}$$

6a.
$$\begin{array}{ccccccccccccccc} & \frac{1}{7} & & \frac{1}{42} & & \frac{1}{105} & & \frac{1}{140} & & \frac{1}{105} & & \frac{1}{42} & & \frac{1}{7} & \\ \frac{1}{8} & & \frac{1}{56} & & \frac{1}{168} & & \frac{1}{280} & & \frac{1}{280} & & \frac{1}{168} & & \frac{1}{56} & & \frac{1}{8} \end{array}$$

6b. Each term of the harmonic triangle is the sum of the two terms below it.

Therefore, $\left\{ {n \atop k} \right\} = \left\{ {n+1 \atop k} \right\} + \left\{ {n+1 \atop k+1} \right\}$.

6c.
$$\left\{ {n \atop 1} \right\} = \frac{1}{n} = \frac{0!}{n}$$
$$\left\{ {n \atop 2} \right\} = \frac{1!}{n(n-1)}$$
$$\left\{ {n \atop 3} \right\} = \frac{2!}{n(n-1)(n-2)}$$
$$\left\{ {n \atop 4} \right\} = \frac{3!}{n(n-1)(n-2)(n-3)}$$

In general, $\left\{ {n \atop k} \right\} = \dfrac{(k-1)!}{n(n-1)(n-2)\ldots(n-k+1)}$ or,

$$\left\{ {n \atop k} \right\} = \frac{(k-1)!}{n(n-1)(n-2)\ldots(n-k+1)} \cdot \frac{(n-k)(n-k-1)(n-k-2)\ldots(3)(2)(1)}{(n-k)(n-k-1)(n-k-2)\ldots(3)(2)(1)}$$

$$\left\{ {n \atop k} \right\} = \frac{(k-1)!(n-k)!}{n!}$$

Also notice the Harmonic Triangle is related to Pascal's Triangle:

$$1 = 1 \div \mathbf{1}$$
$$\tfrac{1}{2} = \tfrac{1}{2} \div \mathbf{1} \quad \tfrac{1}{2} = \tfrac{1}{2} \div \mathbf{1}$$
$$\tfrac{1}{3} = \tfrac{1}{3} \div \mathbf{1} \quad \tfrac{1}{6} = \tfrac{1}{3} \div \mathbf{2} \quad \tfrac{1}{3} = \tfrac{1}{3} \div \mathbf{1}$$
$$\tfrac{1}{4} = \tfrac{1}{4} \div \mathbf{1} \quad \tfrac{1}{12} = \tfrac{1}{4} \div \mathbf{3} \quad \tfrac{1}{12} = \tfrac{1}{4} \div \mathbf{3} \quad \tfrac{1}{4} = \tfrac{1}{4} \div \mathbf{1}$$
$$\tfrac{1}{5} = \tfrac{1}{5} \div \mathbf{1} \quad \tfrac{1}{20} = \tfrac{1}{5} \div \mathbf{4} \quad \tfrac{1}{30} = \tfrac{1}{5} \div \mathbf{6} \quad \tfrac{1}{20} = \tfrac{1}{5} \div \mathbf{4} \quad \tfrac{1}{5} = \tfrac{1}{5} \div \mathbf{1}$$

Therefore, using the binomial coefficient,
$$\left\{ {n \atop k} \right\} = \frac{1}{n} \div \binom{n-1}{k-1}$$
$$= \frac{1}{n} \div \frac{(n-1)!}{(k-1)!(n-k)!}$$
$$= \frac{(k-1)!(n-k)!}{n(n-1)!}$$
$$\mathbf{= \frac{(k-1)!(n-k)!}{n!}}$$

6di. $\displaystyle\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} = \mathbf{\frac{1}{k-1}}$

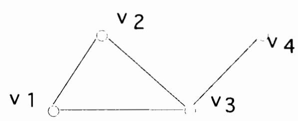
6dii. From 6b, $\left\{ {n \atop k} \right\} = \left\{ {n+1 \atop k} \right\} + \left\{ {n+1 \atop k+1} \right\}$. If $k \geq 2$, n can be replaced with $n-1$ and k can be replaced with $k-1$, to get $\left\{ {n-1 \atop k-1} \right\} = \left\{ {n \atop k-1} \right\} + \left\{ {n \atop k} \right\}$. Therefore, $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} - \left\{ {n \atop k-1} \right\}$.

For $n \geq k \geq 2$, $\displaystyle\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} = \sum_{n=k}^{\infty} \left(\left\{ {n-1 \atop k-1} \right\} - \left\{ {n \atop k-1} \right\} \right)$
$$= \sum_{n=k}^{\infty} \left\{ {n-1 \atop k-1} \right\} - \sum_{n=k}^{\infty} \left\{ {n \atop k-1} \right\}$$
$$= \sum_{n=k}^{k} \left\{ {n-1 \atop k-1} \right\} + \sum_{n=k+1}^{\infty} \left\{ {n-1 \atop k-1} \right\} - \sum_{n=k}^{\infty} \left\{ {n \atop k-1} \right\}$$
$$= \left\{ {k-1 \atop k-1} \right\} + 0$$
$$\mathbf{= \frac{1}{k-1}}.$$

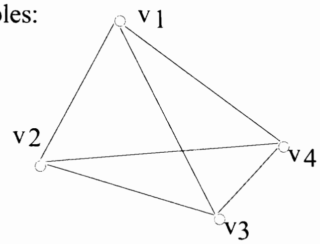
Chromatic Polynomials

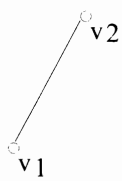
The Definitions and Theorems

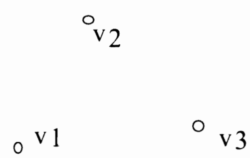
A graph, G, is a finite non empty set V_G of objects called vertices (also called points or nodes) and a (possibly empty) set E_G of two-element subsets of V_G called edges (or lines). Every graph has a diagram associated with it. For example, a graph G defined by $V_G = \{v_1, v_2, v_3, v_4\}$ and $E_G = \{v_1v_2, v_1v_3, v_2v_3, v_3v_4\}$ has the following diagram:



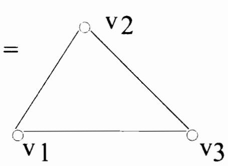
In a null graph (designated N), the set E_G is empty. In a complete graph (designated K), the set E_G contains all possible two-element subsets of V_G. Subscripts are used to indicate the number of elements in V_G. As examples:

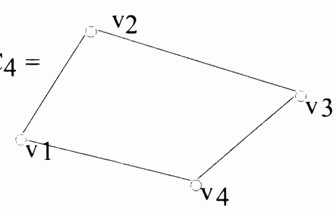
$K_4 =$ 

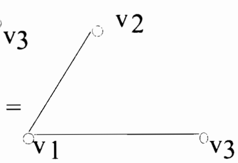
$K_2 =$ 

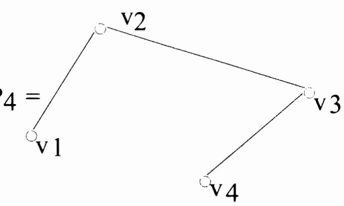
$N_3 =$ 

Other special graphs are cycles (C) and paths (P). As examples :

$C_3 =$ 

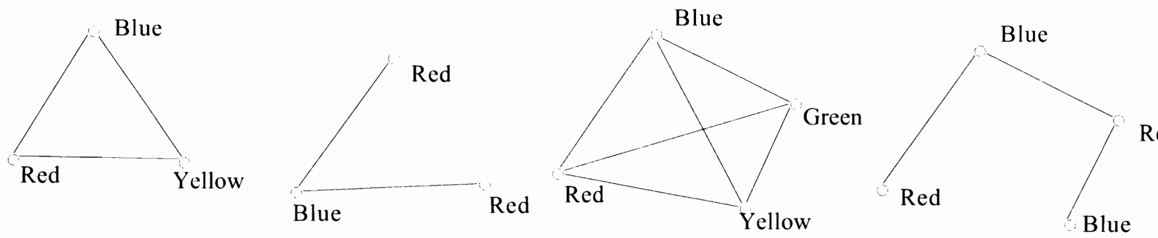
$C_4 =$ 

$P_3 =$ 

$P_4 =$ 

Note that $K_3 = C_3$ and $P_2 = K_2$ and that C_1 and C_2 are undefined.

A graph can be "colored" by assigning a color to each vertex so that vertices connected with a edge are assigned different colors. The minimum number of colors needed to color a graph G is called the chromatic number of the graph, X_G. For example, $X_{K_3} = 3$, $X_{P_3} = 2$, $X_{K_4} = 4$, $X_{P_4} = 2$.



Two colorings of a graph are considered different if they assign different colors to the same vertex of the graph. For example, using two colors, P_3 can be colored in two different ways; however K_3 cannot be colored with only two colors. Using three colors, P_3 can be colored in twelve ways and K_3 can be colored in six ways.

A <u>chromatic polynomial of graph G</u>, $P_G(x)$, is the number of different colorings of G using x colors. For example, using the graphs:

If you have x colors to color G_0, you have x choices for v_1 and also x choices for v_2 and v_3. Therefore, $P_{G_0}(x) = (x)(x)(x) = \mathbf{x^3}$.

If you have x colors to color G_1, you have x choices for v_1, (x – 1) choices for v_2 (it cannot be colored the same as v_1), and x choices for v_3.

Therefore, $P_{G_1}(x) = (x)(x-1)(x) = \mathbf{x^3 - x^2}$.

If you have x colors to color G_2, you have x choices for v_1, (x – 1) choices for v_2 (it cannot be the same as v_1), and (x – 1) choices for v_3 (it cannot be the same as v_2).

Therefore, $P_{G_2}(x) = (x)(x-1)(x-1) = \mathbf{x^3 - 2x^2 + x}$.

If you have x colors to color G3, you have x choices for v_1, (x – 1) choices for v_2, and (x – 2) choices for v_3 (it cannot the same color as v_1 or v_2).

Therefore, $P_{G_3}(x) = (x)(x-1)(x-2) = \mathbf{x^3 - 3x^2 + 2x}$.

Notice that $P_{G_2}(2) = 2$, $P_{G_3}(2) = 0$, $P_{G_2}(3) = 12$, and $P_{G_3}(3) = 6$ are the same results as stated in the paragraph above.

Theorems about Chromatic Polynomials

1. For a null graph, N_n, with exactly n vertices and no edges, $P_{N_n}(x) = x^n$.

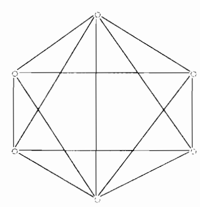
2. For a complete graph, K_n, with exactly n vertices any two of which are connected with an edge,
   $P_{K_n}(x) = x(x-1)(x-2)\ldots(x-n+1)$.

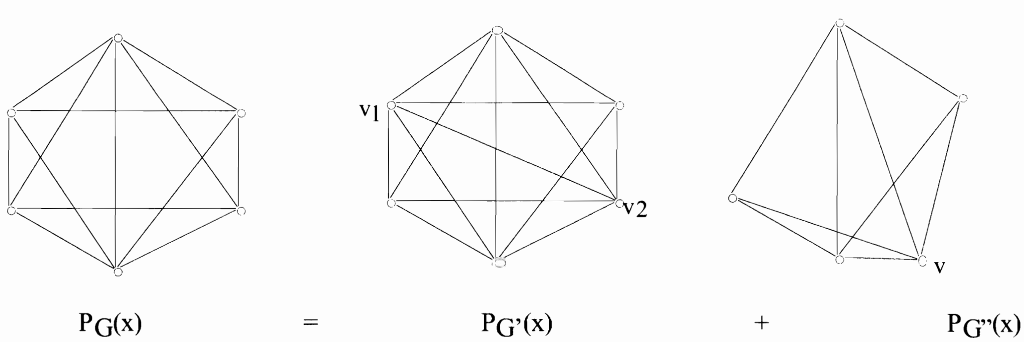
3. For a path of n vertices, P_n, $P_{P_n}(x) = x(x-1)^{n-1}$.

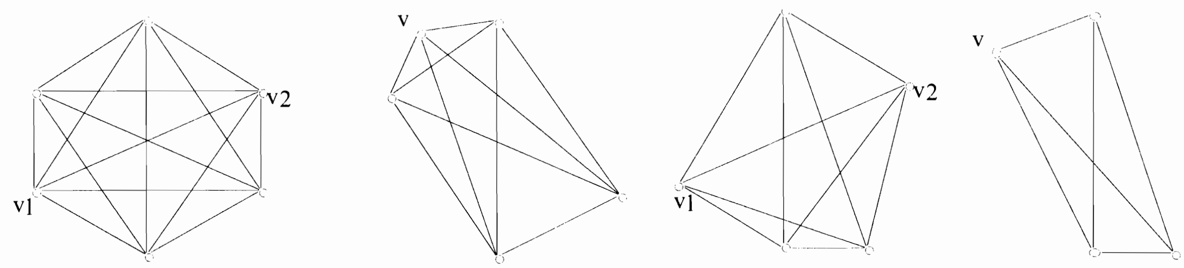
4. If G is a graph with n vertices and m edges, then
   i) $P_G(x)$ has degree n.
   ii) the coefficient of x^n in $P_G(x)$ is 1.
   iii) the coefficient of x^{n-1} in $P_G(x)$ is -m.
   iv) the constant term in $P_G(x)$ is 0.
   v) the coefficients of $P_G(x)$ alternate in sign.
   vi) if $m \neq 0$, then the sum of the coefficients of $P_G(x)$ is 0.

As the graphs become more complex, counting the number of colorings and the choices of colors becomes very difficult and so the following recursive algorithm is very helpful. Repeated use of the algorithm shows for any graph, $P_G(x)$ is the sum the chromatic polynomials of related complete graphs and so Theorem 2 can be applied.

5. Let G be a graph containing two vertices v_1 and v_2 not connected by an edge. From G, form a new graph G' by connecting v_1 and v_2 and then form another graph G" by replacing v_1 and v_2 by a vertex v (not already in G) and connecting v to any vertices which v_1 and v_2 were connected to in G. Then $P_G(x) = P_{G'}(x) + P_{G''}(x)$. The following diagram shows how this theorem can be used three times to find the chromatic polynomial of this graph.







$$= \quad P_{(G')'}(x) \quad + \quad P_{(G')''}(x) \quad + \quad P_{(G'')'}(x) \quad + \quad P_{(G'')''}(x)$$

Therefore, $P_G(x) = P_{(G')'}(x) + P_{(G')''}(x) + P_{(G'')'}(x) + P_{(G'')''}(x)$

$= P_{K6}(x) + P_{K5}(x) + P_{K5}(x) + P_{K4}(x)$

$= (x)(x-1)(x-2)(x-3)(x-4)(x-5) + 2[(x)(x-1)(x-2)(x-3)(x-4)] + (x)(x-1)(x-2)(x-3)$

$= (x)(x-1)(x-2)(x-3)[(x-4)(x-5) + 2(x-4) + 1]$

$= (x)(x-1)(x-2)(x-3)(x2 - 7x + 13)$

$= x^6 - 13x^5 + 66x^4 - 161x^3 + 185x^2 - 78x.$

ARML Power Contest – February 1999 – Chromatic Polynomials

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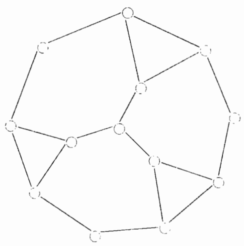
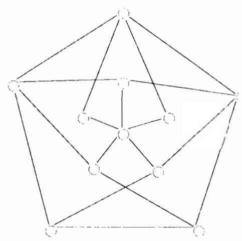
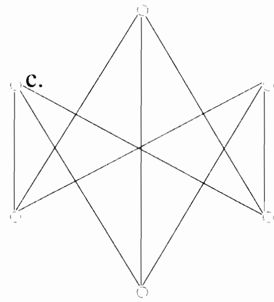
The Problems

1. Find the chromatic number for each of these graphs:

a. 

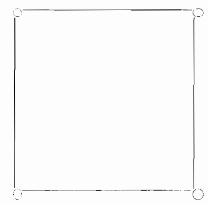
b. 

c. 

2a. Consider graph G_1 at the right.

i) Using two colors, show that G_1 can be colored in two different ways.

ii) Using three colors, show that G_1 can be colored in eighteen different ways.

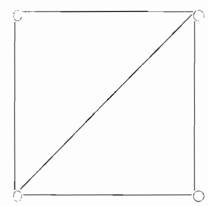


G_1

2b. Consider graph G_2 at the right.

i) Using two colors, in how many different ways can G_2 be colored?

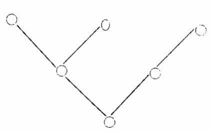
ii) Using three colors, in how many different ways can G_2 be colored?

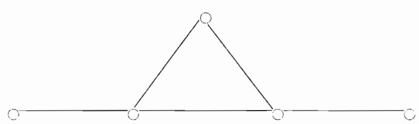


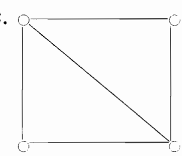
G_2

3. Using the counting method, find $P_G(x)$ for each of these graphs.

(You may leave your answer in factored form.)

a. 

b. 

c. 

4. Determine a suitable graph for each of the following chromatic polynomials:

a. $P_G(x) = x^4 - x^3$

b. $P_G(x) = x^4 - 3x^3 + 3x^2 - x$

c. $P_G(x) = x^4 - 4x^3 + 6x^2 - 3x$

d. $P_G(x) = x^4 - 2x^3 + x^2$

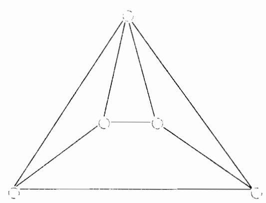
5. a. Prove Theorem 1.

b. Prove Theorem 2.

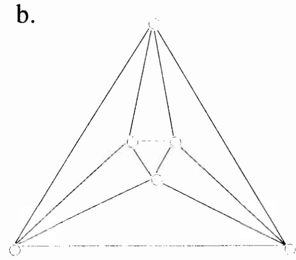
c. Prove Theorem 4 part vi (Hint: How can you find the sum of the coefficients of any polynomial?)

6. Using theorem 5, find $P_G(x)$ for each of these graphs. (You may leave your answer in factored form.)

a.



b.



7. Prove Theorem 5.

The Extensions

Chromatic Polynomials were first used by Birkhoff in 1946 in an attempt to solve the famous Four Color Problem. This problem conjectured only four colors were needed to color any map so countries or regions sharing a common boundary were always colored a different color. It was first proposed in 1852 by Francis Guthrie, a student at the University of London. A proof of the theorem was attempted by many eminent mathematicians, including August de Morgan and Arthur Cayley. Several proofs were published in the 1880's but all were found to contain errors. Interest in the problem grew and spawned many new techniques in mathematics. In Birkhoff's attempt, if a planar graph (a graph which can be drawn in a plane with no edges crossing except at a vertex) could be represented by a chromatic polynomial, then it was only necessary to show that $P_G(4)$ is always at least 1. He was unsuccessful in completing the proof. The Four Color Problem was finally solved in 1976 by Appel and Haken; however, their proof required extensive use of a computer to check the validity of all cases.

Some further problems to consider:

1. If G is a cycle with n vertices, C_n, then $P_{C_n}(x) = (x-1)^n + (-1)^n(x-1)$.
2. If G is a tree with n vertices, T_n, then $P_{T_n}(x) = x(x-1)^{n-1}$.
3. In problems 3a and 5b, you found the chromatic polynomials for the graphs associated with the tetrahedron and the octahedron. What are the chromatic polynomials for the graphs associated with the other Platonic solids - the cube, icosahedron, and dodecahedron?

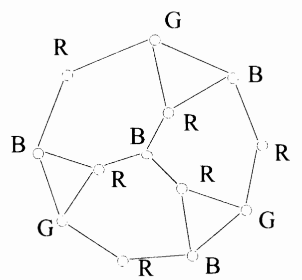
ARML Power Contest – February 1999 – Chromatic Polynomials

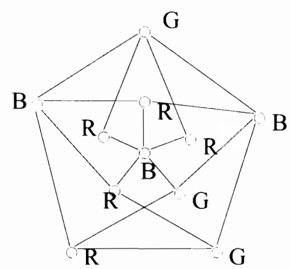
The Solutions

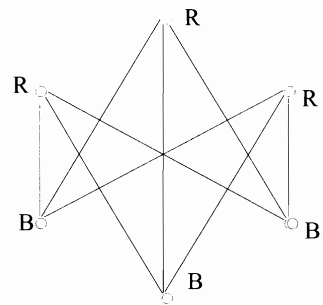
1a. Because the graph contains K_3 as a subgraph, at least three colors are needed. The coloring below shows this is minimal.

1b. The outer pentagon requires at least three colors. The coloring below shows this is minimal.

1c. Since the graph contains an edge, at least two colors are needed. The coloring below shows this is minimal.







2a i. R B   B R
B R   R B

2a ii. R B   R G   R G   R B   B R   B G   B R   B G
B G   G B   B R   G R   R G   G R   G B   R B
G R   G B   G R   G B   R B   R G   B R   B G   G R   G B
R B   B R   B G   R G   B R   G R   R B   G B   R G   B G

2b i. It cannot be colored in two colors

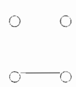
2b ii. six: R B   R G   B R   B G   G R   G B
G R   B R   G B   R B   B G   R G

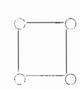
3a. $P(x) = (x)(x-1)^5 = x^6 - 5x^5 + 10x^4 - 10x^3 + 5x^2 - x$

3b. $P(x) = (x)(x-1)^3(x-2) = x^5 - 5x^4 + 9x^3 - 7x^2 + 2x$

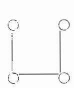
3c. $P(x) = (x)(x-1)(x-2)^2 = x^4 - 5x^3 + 8x^2 - 4x$

4a. $P_G(x) = x^4 - x^3 = x^3(x-1)$ 4c. $P_G(x) = x^4 - 4x^3 + 6x^2 - 3x = x(x-1)(x^2 - 3x + 3)$





4b. $P_G(x) = x^4 - 3x^3 + 3x^2 - x = x(x-1)^3$ 4d. $P_G(x) = x^4 - 2x^3 + x^2 = x^2(x-1)^2$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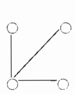

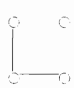
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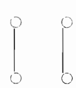


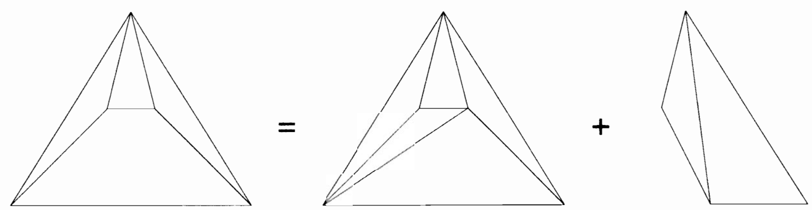
5a. For each of the n vertices you have x choices of colors. Therefore, $P_G(x) = \underbrace{(x)(x)(x)\ldots(x)}_{n\ times} = x^n$.

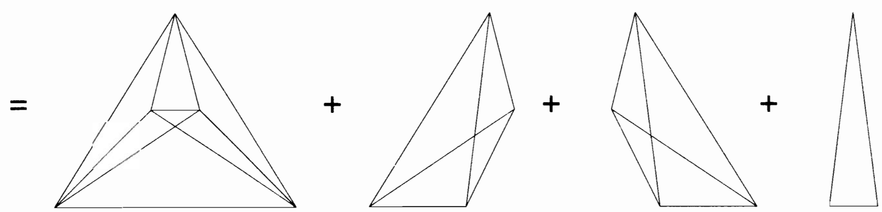
5b. Pick a vertex, v_1; you have x choices of colors for v_1. Pick a second vertex, v_2. Since it is connected to v_1, you only have x – 1 choices of colors for v_2. Pick a third vertex, v_3. Since it is connected to v1 and v_2, you only have x – 2 choices of colors for v_3. Continue this process until there is only one vertex not colored, v_n. Since v_n is connected to all the other n – 1 vertices, it can be colored using x – (n – 1) colors.

Therefore, $P_{K_n}(x) = x(x-1)(x-2)\ldots(x-n+1)$.

5c. Since $m \neq 0$, G has at least one edge. The two vertices connected by this edge, must be colored differently. Therefore, at least two colors are needed to color G and so $P_G(1) = 0$. But in any polynomial, P(1) equals the sum of the coefficients. Therefore, if $m \neq 0$, then the sum of the coefficients of $P_G(x)$ is 0.

6.a.





K_5 K_4 K_4 K_3

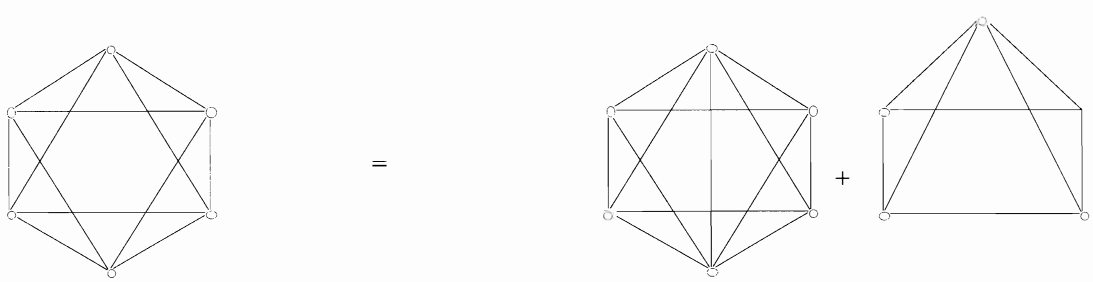
Therefore, $P_G(x) = (x)(x-1)(x-2)(x-3)(x-4) + 2(x)(x-1)(x-2)(x-3) + (x)(x-1)(x-2)$

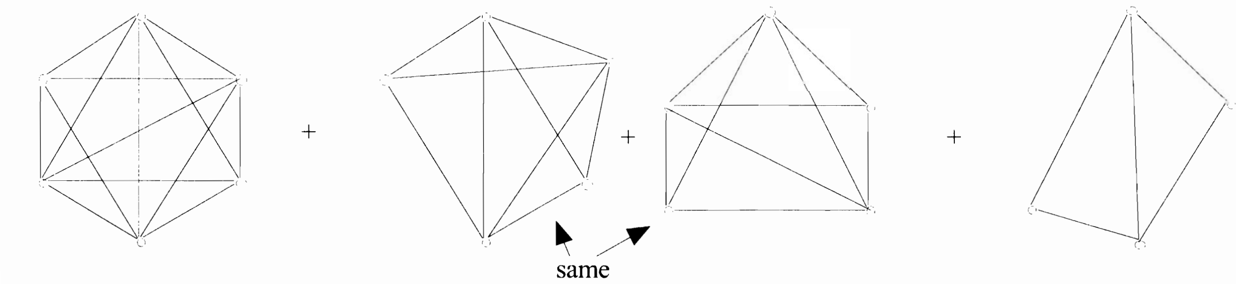
$P_G(x) = (x)(x-1)(x-2)(x^2 - 7x + 12 + 2x - 6 + 1)$

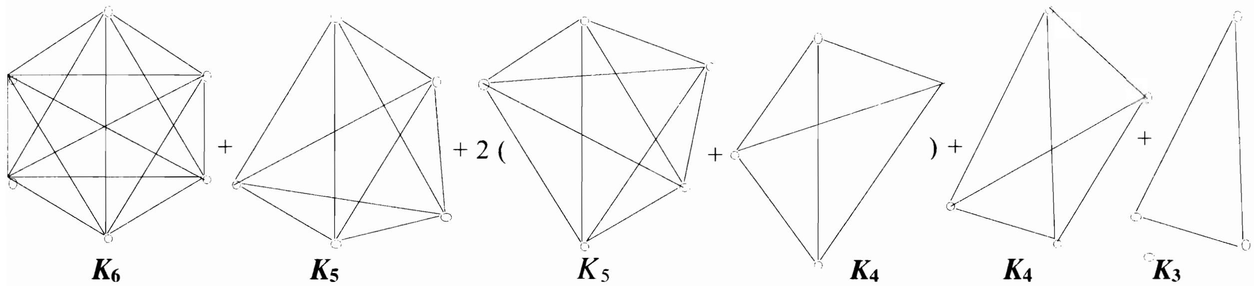
$\mathbf{P_G(x) = (x)(x-1)(x-2)(x^2 - 5x + 7)}$

$\mathbf{P_G(x) = x^5 - 8x^4 + 24x^3 - 31x^2 + 14x}$

6b.







$P_G(x) = (x)(x-1)(x-2)((x-3)(x-4)(x-5) + 3(x-3)(x-4) + 3(x-3) + 1)$

$\boldsymbol{P_G(x) = (x)(x-1)(x-2)(x^3 - 9x^2 + 29x - 32)}$

$\boldsymbol{P_G(x) = x^6 - 12x^5 + 58x^4 - 137x^3 + 154x^2 - 64x}$

7. Consider graph G with two vertices, v_1 and v_2, which are not connected. The number of different colorings of G using x colors is equal to the number of such colorings of G where the color of v_1 is different from v_2 plus the the number of such colorings of G where the colors of v_1 and v_2 are the same. The number of colorings of G where v_1 and v_2 are different is the same as the number of colorings of G', the graph where v_1 and v_2 are connected. The number of colorings of G where v_1 and v_2 are the same color is the same as the number of colorings of G'', the graph where v_1 and v_2 are removed and replaced by a new vertex v with all vertices that were connected to v_1 and v_2 now being connected to vertex v.

Therefore, $P_G(x) = P_{G'}(x) + P_{G''}(x)$.

Twenty-five Point Affine Geometry

The Definitions and Theorems

Definition 1: A point is any letter A through Y. There are twenty-five points in this geometry. The points are arranged in blocks in the following three ways:

A	**B**	**C**	**D**	**E**
F	**G**	**H**	**I**	**J**
K	**L**	**M**	**N**	**O**
P	**Q**	**R**	**S**	**T**
U	**V**	**W**	**X**	**Y**

A	**I**	**L**	**T**	**W**
S	**V**	**E**	**H**	**K**
G	**O**	**R**	**U**	**D**
Y	**C**	**F**	**N**	**Q**
M	**P**	**X**	**B**	**J**

A	**X**	**Q**	**O**	**H**
R	**K**	**I**	**B**	**Y**
J	**C**	**U**	**S**	**L**
V	**T**	**M**	**F**	**D**
N	**G**	**E**	**W**	**P**

Definition 2: A line is any row or column in one of the three blocks above. Therefore, a line contains five distinct points. There are only thirty lines and every point is on six different lines in this finite geometry. For example, THUNB is a line and point T is on lines PQRST, EJOTY, AILTW, THUNB, VTMFD, and XKCTG.

Theorem 1: Given any two distinct points, there is one and only one line containing both of them. For example, given points T and K, only line XKCTG contains both of them.

Definition 3: Two lines are parallel if they have no points in common. The points are arranged in the blocks so that parallel lines must lie in the same block and both be rows or both be columns.

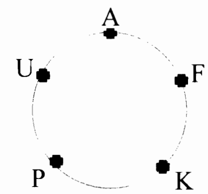
Theorem 2: Given any two distinct lines, either they are parallel or they have only one point in common. Lines FGHIJ and IVOCP both contain point I.

Theorem 3: Given a line and a point not on the line, there is one and only one line containing the point and parallel to the given line. For example, given ASGYM and point R, line LERFX is the only line containing R and parallel to ASGYM.

Definition 4: Two lines are perpendicular if one of them is a row and the other a column in the same block. For example, lines FGHIJ and CHMRW are perpendicular and intersect at point H while lines FGHIJ and LERFX intersect at point F but are not perpendicular.

Theorem 4: Through any point, there is one and only one line perpendicular to a given line. For example, given AFKPU and point N, line KLMNO is perpendicular to line AFKPU and contains N.

Definition 5: The distance between any two points is the least number of steps separating the points on the line which contains them. Lines do not have end points so when considering distances, it might be best to think of the five points of a line as being cyclic: For example, on line AFKPU, distance AF = 1, distance AK = 2, distance AP = 2 (not 3) , and distance AU = 1 (not 4). Row distances do not equal column distances so row distances will be distinguished from column distances by using a prime. Therefore, on line ABCDE, distance AB = 1', AC = 2' , AD = 2' , and AE = 1' . Therefore, between two distinct points there are only four possible distances: 1, 2, 1', and 2'. In this geometry, row distances and column distances are not comparable.



Definition 6 Point x is the midpoint of points **a** and **b** if **x** is on the line containing points a and b and distance ax equals distance xb. For example, on line {EJOTY} point J is the midpoint of points E and O while point T is the midpoint of points E and J.

Two figures are considered different if they have a different set of points.

The Problems

Part A Triangles

Definition: A triangle is a set of three non-collinear points.

1. How many different triangles are there?

2. There are three types of triangles: equilateral, isosceles, and scalene. Each comes in four different sizes. What type of triangle is each of the following: FAR, NHL, and SLO?

3. Prove or Disprove: All scalene triangles are right triangles. (A right triangle has two perpendicular sides.)

4. Find the circumcenter, the centroid, and the orthocenter of triangle ABN. Show these points are collinear. (This line is called the Euler Line.)

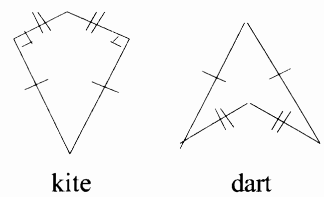
Part B Quadrilaterals

Definition: A quadrilateral is a set of four points, no three of which are collinear.

5. How many different quadrilaterals are there?

6. There are eleven types of quadrilaterals. Match each of these quadrilaterals with its distinct name. (Vertices of a quadrilateral are named in order.)

AVQK	rectangle
AKCR	rhombus
ACQG	general parallelogram (adjacent sides are not $\cong$ or $\perp$)
AKLQ	right trapezoid with 3 congruent sides (!)
MIBF	right trapezoid with 2 congruent sides
VBDW	isosceles trapezoid
BWHL	general trapezoid (no $\perp$'s)
AGQI	kite
BFVH	dart
AYDW	general quadrilateral (no $\parallel$ sides)
APNS	general right trapezoid (no $\cong$ sides)



Once the eleven types of quadrilaterals have been identified, the following theorems can be easily seen. This is NOT part of this Power Contest Question:

The opposite sides of a parallelogram (rectangle, and rhombus) are congruent.

The diagonals of a parallelogram (rectangle, and rhombus) bisect each other.

The diagonals of a rectangle (and isosceles trapezoid) are congruent.

The diagonals of a rhombus (kite, and dart) are perpendicular.

The diagonals of a right trapezoid with three congruent sides are parallel !

7. Prove or disprove: If the opposite sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

8. Prove or disprove: There are no squares in this geometry.

<u>Part C Circles</u>

Definition: A <u>circle</u> is the set of six points a given distance (called the <u>radius</u>) from a given point (called the <u>center</u>).

9a. Find a circle centered at point E with a radius of 2'.

9b. What is the center and radius of the circle {K, J, Y, P, V, G}?

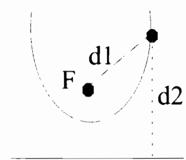
10. How many different circles are there?

11. In any triangle, the three midpoints of the sides, the three feet of the altitudes, and the midpoints of the segments joining the vertices to the centroid all lie on a circle called the Feuerbach Circle. Consider again the triangle ABN from problem 4 in Part A. What is the center of its Feuerbach Circle?

12. Prove or Disprove: A line through the center of a circle always intersects the circle at two points.

Part D Conics

Definition: A parabola is the set of points equidistant from a given point (called the focus) and a given line (called the directrix.)

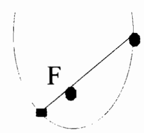


directrix

13. How many different parabolas are there?

14. Determine the set of five points of the parabola whose directrix is line AFKPU and focus is point M.

Definition: A line tangent to a parabola intersects the parabola at only one point and is not perpendicular to the directrix. Of the five points of a parabola, one is the vertex and the other four form two pairs of endpoints of a focal chord (a chord going through the focus).



focal chord

15a. Consider parabola {TIGED} whose focus is point X and directrix is line {YCFNQ}. Determine the five tangent lines to this parabola.

15b. Make a conjecture regarding the tangent line through the vertex. Support your conjecture with evidence.

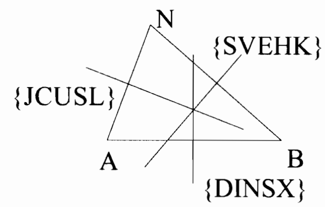
15c. Make a conjecture about a pair tangent lines that are tangent to the endpoints of a focal chord. Your conjecture should involve the directrix. Support your conjecture with evidence.

The Solutions

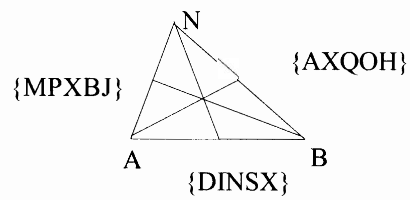
1. $\frac{(25)(24)(20)}{3!} = 2000$

2. FAR = equilateral; NHL = isosceles; SLO = scalene

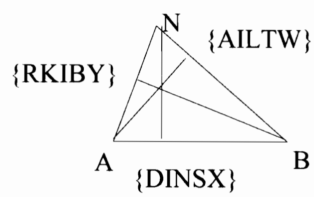
3. For a triangle to be scalene all sides must have different lengths. Since there are only two row distances and only two column distances, one side of the triangle must have a row distance length and another side must have a column distance length. Since the lines containing these sides must intersect, they must lie in the same block with one a row and the other a column. Therefore, these two sides are perpendicular and hence the triangle must be a right triangle.



4a. The circumcenter is the intersection of the perpendicular bisectors of the sides of the triangle. Therefore, the circumcenter is S.



4b. The centroid is the intersection of the medians of a triangle. Therefore the centroid is X.



4c. The orthocenter is the intersection of the altitudes of a triangle. Therefore, the orthocenter is I.

Points S, X, and I all lie in line {DINSX}, the Euler Line of this triangle.

5. $\frac{(25)(24)(20)(13)}{4!} = 6500$

6. AVQK: AK || VQ and AV = QK. Therefore, AVQK is a isosceles trapezoid.

AKCR: AK || RC, AR || KC, no perpendiculars. AKCR is a general parallelogram.

ACQG: AC = CQ, QG = GA, AC $\perp$ QG at B, CQ $\perp$ GA at Y. ACQG is a dart.

AKLQ: AK || QL, AK $\perp$ KL, KL $\perp$ QL, no opposite congruent sides. Therefore, AKQL is a general right trapezoid.

MIBF: MI || BF, IB || MF, IB $\perp$ IM. Therefore, MIBF is a rectangle.

AGVF: GV || FA, AG $\neq$ VF, no perpendiculars. Therefore, AGVF is a general trapezoid.

VBDW: VW || BD, VB $\perp$ WB, VB $\perp$ BD, no congruent sides. Therefore, VBDW is a general right trapezoid.

BWHL: BW || HL,WH || LB, BW = WH = HL = LB, no perpendiculars. Therefore, BWHL is a rhombus.

AGQI: no parallels, AG = GQ, QI $\neq$ IA. Therefore, AGQI is a general quadrilateral.

BFVH: no parallels, BF = HB, FV = VH, HB $\perp$ VB, VH $\perp$ HB. Therefore, BFVH is a kite.

AYDW: AY || DW, DW $\perp$ WA $\perp$ AY, AY = YD = DW $\neq$ WA. Therefore, AYDW is a right trapezoid with three congruent sides.

APNS: AP || NS, no perpendiculars. Therefore , APNS is a general trapezoid.

7. False. Quadrilateral AYDW is a counter example. AY || DW and AY = DW = 2 but YD is not || WA. Therefore AYDW is not a parallelogram.

8. True. A square is a rectangle with congruent sides. Therefore, a square must have perpendicular and congruent adjacent sides. However, this is impossible since perpendicular sides must lie in the same block, one in a row and the other in a column but row lengths are never equal to column lengths.

9a. {BKNCSP}

9b. center = A, radius = 2. Consider the midpoints of the five chords with endpoint K, the midpoints of the five chords with endpoint J, and the midpoints of the five chords with endpoint Y. These sets have only point A in common. (Or alternatively, the perpendicular bisector of chord KJ is {AILTW} and the perpendicular bisector of chord KY is {ARJVN}. They intersect at the center A.)

10. (25)(4) = 100 .

11. The midpoints of the sides are J, D, and H and the feet of the altitudes are R, T and D. The circle containing these points is centered at N with a radius of 2.

12. False. Circle {HIBXWE} is centered at A with a radius of 1'. However, line {AFKPU} goes through the center A but does not contain any point on the circle.

13. (30)(20) = 600 .

14. 1' from {AFKPU} and 1' from point M = {LJT}; 2' from {AFKPU} and 2' from point M = {DX}. Therefore, the parabola is {LJTDX}.

15a. {PQRST}, {RKIBY}, {BGLQV}, {SVEHK}, and {HYLDP}

15b. The tangent line through the vertex is parallel to the directrix. E must be the vertex since T, X, and G lie on line {XKCTG} and I, X, and D lie on line {DINSX}. {SVENK} is the tangent line through E. Both tangent line {SVENK} and the directrix {YCFNQ} are rows in block 2 and hence they are parallel.

15c. Tangent lines through the endpoints of a focal chord are perpendicu-lar and intersect on the directrix. Points T and G are the endpoints of a focal chord. {PQRST} and {BGLQV} are tangent lines through these points. They are perpendicular in block I and intersect at Q, a point on the directrix {YCFNQ}. (Also, points I and D are the endpoints of a focal chord. {RKIBY} and {HYLDP} are tangent lines through these points. They are perpendicular in block III and intersect at Y, a point on the directrix {YCFNQ}.)

The Extensions

Many more theorems and investigations of this geometry are presented in Puzzles and Paradoxes by T. H. O'Beirne, including its association with a triangular lattice system. Of unusual interest is how this geometry is used to solve a problem regarding the placement of photographic equipment around a circular race track!

Square-Sum Partitions

The Definitions

Consider a set A of distinct elements. A partition of set A is a set of disjoint subsets of A whose union equals set A. For example, the set $\{\{1, 3\}, \{2, 4, 6\}, \{5\}\}$ is a partition of the set $\{1, 2, 3, 4, 5, 6\}$. A square-sum partition of set A is a partition of A in which the elements in each of the subsets in the partition add up to a square number. For example, the set $\{\{0, 1, 6, 9\}, \{2, 7\}, \{3, 5, 8\}, \{4\}\}$ is a square-sum partition of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

For ease in correcting please write all subsets in ascending order and all partitions in ascending according to the first element of each subset as shown in the examples.

The Problems

Part I Square-Sum Pairs Consider the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. It can be partitioned into square-sum pairs: $\{\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4,5\}\}$ in which each subset contains a pair of elements that add up to a square number. We will now investigate the partitioning of the set $\{0, 1, 2, 3, \ldots, n\}$ into square-sum pairs.

1. Partition the sets $\{0, 1, 2, 3, \ldots, n\}$ into square-sum pairs where:
   a) $n = 1$ b) $n = 7$ c) $n = 9$

2. Prove: If n is the square of an odd number, then $\{0, 1, 2, 3, \ldots, n\}$ can be partitioned into square-sum pairs.

3. Prove the sets $\{0, 1, 2, 3, \ldots, n\}$ cannot be partitioned into square-sum pairs when:
   a) $n = 3$ b) $n = 5$ c) $n = 11$

4. Sometimes there exists a number, S, the square of an odd number less than $2n$, that can be used to partition $\{0, 1, 2, 3, \ldots, n\}$ into two sets $\{0, 1, 2, 3, \ldots, S - n - 1\}$ and $\{S - n, S - n + 1, S - n + 2, \ldots, n - 1, n\}$ where the first set, $\{0, 1, 2, 3, \ldots, S - n - 1\}$, is already known to be partitionable into square-sum pairs and the second set, $\{S - n, S - n + 1, S - n + 2, \ldots, n - 1, n\}$, is easily shown to be partitionable into square-sum pairs. For example, given the set $\{0, 1, 2, 3, \ldots, 15\}$ and $S = 25$, then $S - n = 10$. $\{0, 1, 2, 3, \ldots, 15\}$ can be partitioned into sets $\{0, 1, 2, 3, \ldots, 9\}$ and $\{10, 11, 12, 13, 14, 15\}$. In exercise 1 you demonstrated $\{0, 1, 2, 3, \ldots, 9\}$ is partitionable into square-sum pairs and $\{10, 11, 12, 13, 14, 15\}$ can be partitioned into $\{10, 15\}$, $\{11, 14\}$, and $\{12, 13\}$, all of which are square-sum pairs. Use this technique to partition $\{0, 1, 2, 3, \ldots, n\}$ into square-sum pairs where:
   a) $n = 17$ b) $n = 23$ c) $n = 27$ d) $n = 29$ e) $n = 31$ f) $n = 33$ g) $n = 35$

5. Sometimes the above technique does not work. Partition the set $\{0, 1, 2, 3, ..., n\}$ into square-sum pairs where:
   a) $n = 13$ b) $n = 19$ c) $n = 21$ d) $n = 37$ e) $n = 39$.

Theorem: For any odd number, $n \geq 41$, there exists an odd square number between $n + 13$ and $2n$.

6. Use the above theorem and the results so far to prove the set $\{0, 1, 2, 3, \ldots, n\}$ can be partitioned into square-sum pairs for any odd n except $n = 3$, 5, and 11.

Part II Square-Sum Triples Can the set $\{0, 1, 2, 3, \ldots, n\}$, where n is one less than a multiple of three, be partitioned into subsets of three numbers which add up to a square number? It is impossible for the set $\{0, 1, 2\}$.

7. Prove it is impossible to partition the set $\{0, 1, 2, 3, 4, 5\}$ into square-sum triples.

8. Show $\{0, 1, 2, ..., 8\}$ can be partitioned into square-sum triples.

9. Show $\{0, 1, 2, ..., 11\}$ can be partitioned into square-sum triples.

The Extensions (not part of the contest)

1. $\{0, 1, 2, ..., 17\}$ can be partitioned into square-sum triples, $\{\{0, 3, 6\}, \{1, 7, 8\}, \{2, 5, 9\}, \{4, 10, 11\}\}$ but $\{0, 1, 2, ..., 14\}$ cannot be partitioned into square-sum triples. Can you prove this?

2. Does there exist a number k, such that if $n \geq k$, it is always possible to partition $\{0, 1, 2, 3, ..., 3n-1\}$ into square-sum triples?

3. Can you prove the theorem used in problem 6?

4. Using the Bertrand-Cebysev Theorem [1850] which states for any $n > 3$, there exists a prime number between n and $2n$, it can be proven, for any even number, n, it is possible to partition the set $\{1, 2, 3, ..., n\}$ into prime-sum pairs.

5. $\{0, 1, 2, ..., 15\}$ can be partitioned into square-sum pairs in two different ways. Compare your answer to 4a and the following set $\{\{0,4\}, \{1,8\}, \{2,14\}, \{3,6\}, \{5,11\}, \{7,9\}, \{10,15\}, \{12,13\}\}$. The set $\{0, 1, 2, ..., 17\}$ can be partitioned into both square-sum pairs and square-sum triples (See 4b and Ext. 1). Can you write a computer programs to find all the square-sum partitions of the set $\{0, 1, 2, 3, ..., n\}$ for a particular odd n?

The Solutions

1a. { {0, 1} } 1b. { { 0, 1}, {2, 7}, {3, 6}, {4, 5} } 1c. { {0, 9}, {1, 8}, {2, 7}, {3, 6}, {4, 5} }

2. If n is the square of an odd number then {0, 1, 2, 3, ..., n} can be partitioned as { {0, n}, {1, n - 1}, {2, n - 2}... {(n - 1)/2, (n + 1)/2} }, where each of the n/2 pairs add up to n, a square number.

3a. No other number of the set can be paired with 2 to add up to a square.

3b. The only number that can be paired with 5 is 4, leaving the set {0, 1, 2, 3} which cannot be not partitioned into square-sum pairs. Or again, no other number of the set can be paired with 2 to add up to a square.

3c. {11, 5} and {10, 6} must be pairs since a sum of 25 cannot be reached with this set. 9 can be paired with 0 or 7, but 2 must be paired with 7 and so 9 must be paired with 0. This leaves {1, 3, 4, 8} but 4 cannot be paired with any of the remaining numbers to form a square.

4a. S = 25. {0, 1, 2, ..., 17} = { {17, 8}, {16, 9}, {15, 10}, {14, 11}, {13, 12}, {0, 1, 2, ... , 7} } where {0, 1, 2, ..., 7} was partitioned in problem 1b.

4b. S = 25. {0, 1, 2, ..., 23} = { {23, 2}, {22, 3}, {21, 4}, {20, 5}, {19, 6}, {18, 7}, {17, 8}, {16, 9}, {15, 10}, {14, 11}, {13, 12}, {0, 1} } where {0,1} was partitioned in problem 1a.

4c. S = 49. {0, 1, 2, ..., 27} = { {27, 22}, {26, 23}, {25, 24}, {0, 1, 2, ... , 23} } where {0, 1, 2, ..., 23} was partitioned in problem 4b.

4d. S = 49. {0, 1, 2, ..., 29} = { {29, 20}, {28, 21}, {27, 22}, {26, 23}, {25, 24}, {0, 1, 2, ..., 19} } where {0,1,2, ..., 19} is partitioned in problem 5b.

4e. S = 49. {0, 1, 2, ..., 31} = { {31, 18}, {30, 19}, {29, 20}, {28, 21}, {27, 22}, {26, 23}, {25, 24}, {0, 1, 2, ..., 17} } where {0, 1, 2, ..., 17} was partitioned in problem 4a.

4f. S = 49. {0, 1, 2, ..., 33} = { {33, 16}, {32, 17}, {31, 18}, {30, 19}, {29, 20}, {28, 21}, {27, 22}, {26, 23}, {25, 24}, {0, 1, 2, ..., 15} } where {0, 1, 2, ..., 15} was partitioned in problem 4 example.

4g. S = 49. {0, 1, 2, ..., 35} = {{35, 14}, {34, 15}, {33, 16}, {32, 17}, {31, 18}, {30, 19}, {29, 20}, {28, 21}, {27, 22}, {26, 23}, {25, 24}, {0, 1, 2, ..., 13}} where {0, 1, 2, ..., 13} is partitioned in problem 5a.

5a. {0, 1, 2, ..., 13} = {{0, 9}, {1, 8}, {2, 7}, {3, 13}, {4, 12}, {5, 11}, {6, 10}}

5b. Three possible answers: {0, 1, 2, ..., 19} = {{0, 1}, {2, 14}, {3, 13}, {4, 12}, {5, 11}, {6, 19}, {7, 18}, {8, 17}, {9, 16}, {10, 15}} = {{0, 4}, {1, 3}, {2, 14}, {5, 11}, {6, 19}, {7, 18}, {8, 17}, {9, 16}, {10, 15}, {12, 13}} = {{0, 4}, {1, 8}, {2, 14}, {3, 6}, {5, 11}, {7, 18}, {9, 16}, {10, 15}, {12, 13}, {17, 19}}

5c Four possible answers: {0, 1, 2, ..., 21} = {{0, 9}, {1, 3}, {2, 14}, {4, 21}, {5, 11}, {6, 19}, {7, 18}, {8, 17}, {10, 15}, {12, 13}, {16, 20}} = {{0, 9}, {1, 8}, {2, 14}, {3, 13}, {4, 12}, {5, 11}, {6, 10}, {7, 18}, {15, 21}, {16, 20}, {17, 19}} = {{0, 16}, {1, 8}, {2, 14}, {3, 13}, {4, 12}, {5, 11}, {6, 10}, {7, 9}, {15, 21}, {16, 20}, {17, 19}} = {{0, 9}, {1 ,8}, {2, 14}, {3, 6}, {4, 21}, {5, 11}, {7, 18}, {10, 15}, {12, 13}, {16, 20}, {17, 19}}

5d. {0, 1, 2, ..., 37} = {{0, 1}, {2, 7}, {3, 22}, {4, 21}, {5, 11}, {6, 10}, {8, 28}, {9, 27}, {12, 37}, {13, 36}, {14, 35}, {15, 34}, {16, 33}, {17, 32}, {18, 31}, {19, 30}, {20, 29}, {23, 26}, {24, 25}}

5e. {0, 1, 2, ..., 39} = {{0, 1}, {2, 34}, {3, 22}, {4, 32}, {5, 31}, {6 ,30}, {7, 18}, {8, 28}, {9, 27}, {10, 39}, {11, 38}, {12, 37}, {13, 36}, {14, 35}, {15, 21}, {16, 33}, {17, 19}, {20, 29}, {23, 26}, {24, 25}}

6. In questions 1, 4, and 5 above, it was demonstrated that for n = odd numbers 1, 7, 9, 11 through 39, the set {0, 1, 2, ..., n} can be partitioned into square-sum pairs. It must be now shown that for $n \geq 41$, the set {0, 1, 2, ..., n} is also partitionable into square-sum pairs. Proof by Induction: Assume for some $k \geq 39$ where k is odd, {0, 1, 2, ..., k} can be partitioned into square-sum pairs, prove {0, 1, 2, ..., k, $k + 1$, $k + 2$} can also be partitioned into square-sum pairs. By the theorem above, for $n \geq 41$ there exists an odd square, s, such that $n + 13 < s < 2n$. Replacing n with $k + 2$, implies that there exists an odd square, S, such that $k + 15 < S < 2k + 4$.

Since $k + 15 < S < 2k + 4$, $13 < S - k - 2 < k + 2$. Therefore, the set {0, 1, 2, ..., k, $k + 1$, $k + 2$} can be partitioned into two sets: {0, 1, 2, ..., $S - k - 3$} and {$S - k - 2$, $S - k - 1$, ... , $k + 1$, $k + 2$}. Since

S – k – 3 is an odd number ≥ 13, the induction hypothesis states the set, {0, 1, 2, ..., S - k - 3}, is partitionable into square-sum pairs and the set {S – k – 2, S – k –1, ... , k + 1, k + 2} = {{S – k – 2, k + 2}, {S – k – 1, k + 1}, ... , {(S – 1) / 2, (S + 1) / 2}}, where all pairs add up to S.

7. The numbers in the set sum to 15, so you need two squares that sum to 15. The smallest possible square triple sum is 4 and the largest would be 9. However, using these square numbers, the only sums possible are 8, 13, and 18. Therefore, since there do not exist two squares that add up to 15, {0, 1, 2, 3, 4, 5} cannot be partitioned into square-sum triples.

8. {0, 1, 2, 3, 4, 5, 6, 7, 8} = {{0, 1, 3}, {2, 6, 8}, {4, 5, 7}}

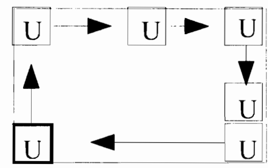
9. Twelve possible answers! : {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} = {{0, 3, 6}, {1, 7, 8}, {2, 5, 9}, {4, 10, 11}} = {{0, 3, 6}, {1, 7, 8}, {2, 4, 10}, {5 ,9, 11}} = {{0, 6, 10}, {1, 7, 8}, {2, 3, 4}, {5, 9, 11}} = {{0, 7, 9}, {1, 3, 5}, {2 ,6, 8}, {4, 10, 11}} = {{0, 1, 8}, {2, 5, 9}, {3 ,6, 7}, {4, 10, 11}} = {{0, 2, 7}, {1, 4, 11}, {3, 5, 8}, {6, 9, 10}} = {{0, 2, 7}, {1, 5, 10}, {3, 4, 9}, {6, 8, 11}} = {{0, 7, 9}, {1, 3, 5}, {2, 4, 10}, {6, 8, 11}} = {{0, 2, 7, {1, 6, 9}, {3, 5, 8}, {4, 10, 11}} = {{0, 5, 11}, {1, 2, 6}, {3, 4, 9}, {7, 8, 10}} = {{0, 3, 6}, {1, 4, 11}, {2, 5, 9}, {7, 8, 10}} = {{0, 4, 5}, {1, 7, 8}, {2, 3, 11}, {6, 9, 10}}

Slides, Rolls, and Rolides

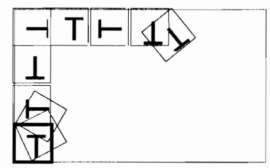
The Definitions

In this problem you will be investigating various polygons and circles moving clockwise inside a rectangle always staying in contact with the rectangle. Three different movements will be considered:

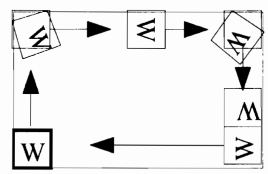
Slides – In this movement, the orientation of the polygon will not change. The polygon will move up, then right, down, and then left around the rectangle, again always staying in contact with the rectangle.



Rolls – In this movement, the polygon will rotate about a vertex which is in contact with the rectangle, "rolling around" the inside of the rectangle. The orientation of the polygon will be constantly changing. The rotation of the polygon will always be counter-clockwise, enabling the polygon to roll clockwise around the rectangle. A circle will roll "without slipping."



Rolides – This movement is a combination of slides and rotations. In this movement the polygon slides into the next corner and then rotates 90° clockwise, always keeping two points of the polygon in contact with the rectangle, and then slides into the next corner and rotates again. In this movement, a polygon will make one complete (360°) rotation in one trip around the rectangle.



In a trip around the rectangle, whether by sliding, rolling, or roliding, the polygon must come back to its original position and original orientation.

The Problems

Part A – Slides

1. Consider a rectangle ABCD "sliding around" inside a rectangle, prove that the distance traveled by vertex A is equivalent for both starting positions shown in the figures below.

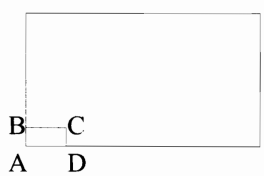


fig.1a

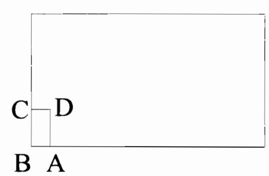


fig.1b

2. Consider a right triangle ABC, with CA = 3, CB = 4 and AB = 5, "sliding around" inside an 9 by 12 rectangle. How much further does vertex C travel in one trip around the rectangle, if it starts in the position in figure 2a versus if it starts in the position in figure 2b?

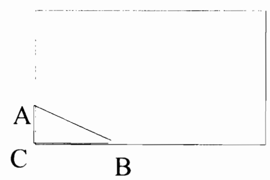


fig.2a

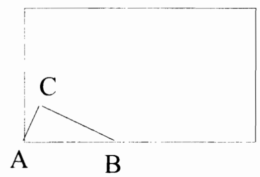


fig.2b

3. Consider right triangle ABC, with integer length sides and right angle at C, "sliding around" inside a rectangle. When positioned as in figure 2a above, the distance traveled by point C in a trip around the rectangle is 12 units longer than the distance traveled by C when the triangle is positioned as in figure 2b. Determine the lengths of the sides of triangle ABC.

Part B – Rolls

Consider a unit square ABCD "rolling around" inside an 9 by 13 rectangle as in figure 4.

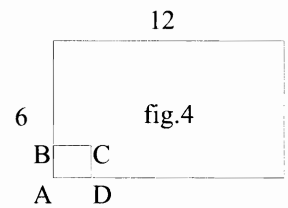


fig.4

4a. How far does vertex A travel in one trip around the rectangle?

4b. How far does the center of the square travel in one trip around the rectangle?

Consider rectangle ABCD, with AB = 1 and AD = 2, "rolling around" inside a 6 by 12 rectangle.

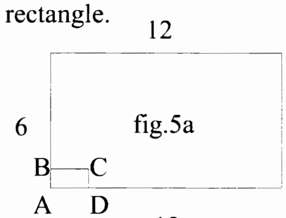


fig.5a

5a. How far does vertex A travel in one trip around the rectangle, if it starts in the position in figure 5a?

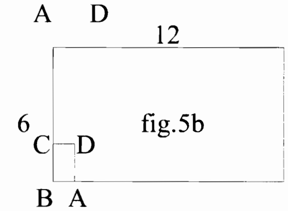


fig.5b

5b. How far does vertex A travel in one trip around the rectangle, if it starts in the position in figure 5b

Consider a circle whose diameter is 1 "rolling around" inside a (π + 1) by (3π+1) rectangle.

6a. How far does the center travel in one trip around the rectangle?

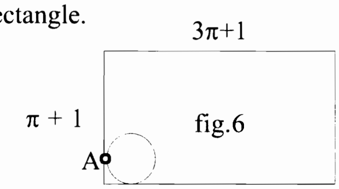


fig.6

6b. How far does tangent point A travel in one trip around the rectangle, if it starts in the position in figure 3? (Hint: "The length of a cycloid, from cusp to cusp, is 4 times the diameter of the generating circle." Sir Christopher Wren, 1658)

Part C – Rolides

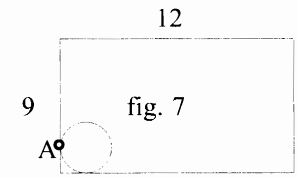


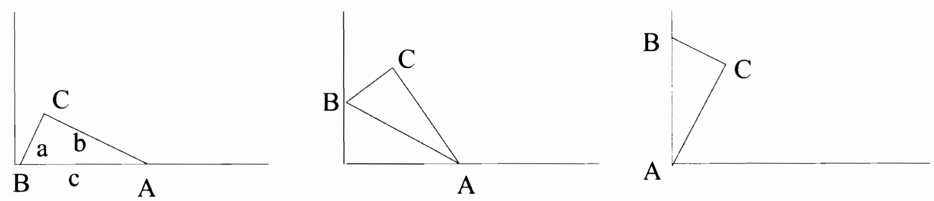
fig. 7

7. Consider a circle whose diameter is 1 "roliding around" inside a 9 by 12 rectangle. How much farther does tangent point A travel than does the center travel in one trip around the rectangle?

Consider a unit square ABCD (with point A starting in the corner), "roliding around" inside a 9 by 12 rectangle.

8a. How far does point A travel in one trip around the rectangle?
8b. How far does the center of the square travel in one trip around the rectangle?

<u>Theorem</u>: Consider a right triangle ABC with right angle at C "roliding around" inside a rectangle. As segment AB moves around any corner, from position 1 to position 3, the locus of points traced out by vertex C is a line segment whose length is 2c – a – b.



before the rotating | during the rotating | after the rotating

9. Consider a 3-4-5 right triangle with right angle at C "roliding around" inside a 9 by 12 rectangle, starting as in position 1 above, how far does vertex C travel in one trip around the rectangle?

10. A right triangle, starting as in position 1 above, "rolides around" inside a rectangle. If $b = a + 1$ and vertex C travels 48 cm less than either vertex A or B, what are the lengths of the three sides of triangle ABC?

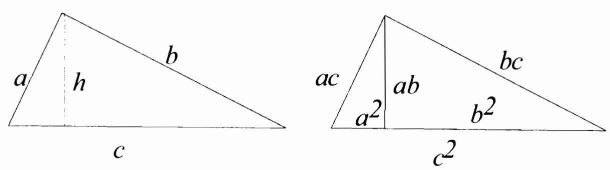
11. Prove the above theorem.

The Solutions

1. Let W and L be the length and width of the large rectangle and $w = AB$ and $l = BC$. The distance traveled by point A in fig.1a is $2(W - w) + 2(L - l)$ and the distance traveled by point A in fig.1b is $2(W - l) + 2(L - w)$. Both when expanded equal $2W + 2L - 2w - 2l$.

2. In fig.2a the distance traveled is $6 + 8 + 6 + 8 = 28$. In fig.2b, using similar triangles, the altitude from vertex C is 2.4. Therefore, the distance traveled is $6.6 + 7 + 6.6 + 7 = 27.2$. Therefore, the difference is 0.8

3. The distance traveled in Fig.2a is $D_1 = 2(W - a + L - b)$. The distance traveled in Fig.2b is $D_2 = 2(W - h + L - c)$. The difference is $12 = D_1 - D_2 = 2(h + c - a - b)$. Therefore, $6 = h + c - a - b$. Since $a, b, and\ c$ are integers, h must be an integer. Using the area formula for the triangle, $\frac{1}{2}(base)(height)$, $\frac{1}{2}ab = \frac{1}{2}hc$ and so $h = \frac{ab}{c}$, i.e., h must divide ab. If $\{a, b, c\}$ is a primitive Pythagorean triple, i.e., a Pythagorean triple with no common factors, h will not divide ab. To produce the desired results, multiply the sides by c.



Looking at common primitive Pythagorean triples, the following table is produced:

a	b	c	ac	bc	c^2	ab	$ab + c^2 - ac - bc$
3	4	5	15	20	25	12	2
5	12	13	65	156	169	60	8
8	15	17	136	255	289	120	18
7	24	25	175	600	625	168	18
20	21	29	580	609	841	420	72

Since in our problem $ab + c^2 - ac - bc = 6$, the only possible triangle is a multiple of the 15, 20, 25 right triangle. To produce $ab + c^2 - ac - bc = 6$, a triangle with side lengths $45, 60, 75$ would be needed.

4a. On each roll the center will travel $\frac{\sqrt{2}\pi}{4}$ units. There will be a total of 40 rolls and so the center will travel $10\sqrt{2}\,\pi$ units.

4b. When the square "rolls" on vertex B, vertex A traces out a circular arc whose length ($\frac{C}{4} = \frac{\pi r}{2}$) is equal to $\frac{\pi}{2}$;

when the square "rolls" on vertex C, vertex A traces out a circular arc whose length is equal to $\frac{\sqrt{2}\pi}{2}$; when the square "rolls" on vertex D, vertex A traces out a circular arc whose length is equal to $\frac{\pi}{2}$; and when the square "rolls" on vertex A, vertex A does not move. Starting with A in the lower-left corner, vertex B ends up in the upper-left corner, vertex C in the upper right, and D on the lower right corner. There will be a total of ten rolls on each vertex. Therefore, vertex A will travel $10(\frac{\sqrt{2}\pi}{2}) + 20(\frac{\pi}{2}) = (5\sqrt{2} + 10)\pi$ units.

5. Again using arc length = $\frac{C}{4} = \frac{\pi r}{2}$, when the square "rolls" on vertex B, vertex A traces out a circular arc whose length is equal to $\frac{\pi}{2}$; when the square "rolls" on vertex C, vertex A traces out a circular arc whose length is equal to $\frac{\sqrt{5}\pi}{2}$; when the square "rolls" on vertex D, vertex A traces out a circular arc whose length is equal to π; and when the square "rolls" on vertex A, vertex A does not move. Therefore, the total distance vertex A travels in fig.5a is $6(\frac{\pi}{2}) + 6(\frac{\sqrt{5}\pi}{2}) + 6(\pi) = (9 + 3\sqrt{5})\pi$ and the distance it travels in fig.5b is

$2(\frac{\pi}{2}) + 6(\frac{\sqrt{5}\pi}{2}) + 6(\pi) = (7 + 3\sqrt{5})\pi$.

6a. Ans: 8π

6b. Ans: 32

7. Ans: π When the circle slides from corner to corner, the center and vertex A travel the same distance. When turning in each corner the center is stationary while vertex A travels a quarter of the circumference or $\frac{\pi}{4}$.

8a. Ans: 42. When roliding around the rectangle, vertex A is always in contact with the rectangle and never needs to backtrack.

8b. Ans: $46 - 4\sqrt{2}$. When sliding inside the rectangle, the center of the square will travel on an 8 by 11 rectangle and when the square rotates around a corner, it travels $2 - \sqrt{2}$ units. Therefore, the total distance is $2(8 + 11) + 4(2 - \sqrt{2}) = 38 + 8 + 4\sqrt{2}$.

9. Ans: 34. Because side AB is always in contact with the rectangle when the triangle is sliding, vertex C travels $2(9 - 5 + 12 - 5) = 22$ units. While the triangle rotates in each corner, vertex C travels $2(5) - 3 - 4 = 3$, Therefore, the total distance traveled is $22 + 4(3) = 34$ units.

10. When ΔABC rolides around a rectangle with dimensions $L\ by\ W$, vertex A travels $2(L+W)$ units, while vertex C travels $2(W-c+L-c)+4(2c-a-b)$ units.
    Therefore, $2W+2L-(2W-2c+2L-2c+8c-4a-4b)=48$, which simplifies to $a+b-c=12$. Since $b=a+1$, $a+b-c=12$ reduces to $2a-c=11$ or $c=2a-11$. Substituting these equations into $a^2+b^2=c^2$ yields $a^2+(a+1)^2=(2a-11)^2$, which simplifies to $a^2-23a+60=(a-20)(a-3)=0$. And so or . If $a=3$ then ΔABC has sides of lengths $3, 4, \text{and} -5$, an impossibility. Therefore, $a=20$ and the lengths of the sides are $20, 21, \text{and } 29$.

11. Part i) Consider right ΔABC roliding around a corner in the general position in figure 11a. Since both triangles ABC and ABO are right triangles with hypotenuse AB, $ABCO$ is a cyclic quadrilateral (figure 11b). Draw ray $\overrightarrow{OC}$ (figure 11c). $\angle COA$ is always equal to $\angle B$ (of ΔABC) since both angles intercept arc $\overset{\frown}{AC}$. Therefore, vertex C will always travel on a line when roliding around a corner.
    Part ii) When right ΔABC rolides into a corner, vertex B is in the corner and vertex C is a units from the corner (figure 11d). As ΔABC rolides it eventually gets to the position in figure 11e, with the legs parallel to the sides of the rectangle. Now vertex C is c units from the corner and hence has traveled $c-a$ units in going from figure 11d to figure 11e. As ΔABC continues to rolide, eventually it gets to the position in figure 11f and is ready to leave the corner. Vertex C is now b units from the corner and so in going from figure 11e to figure 11f, vertex C has traveled $c-b$ units. Therefore, vertex C will travel a total of $2c-a-b$ units when roliding around a corner.

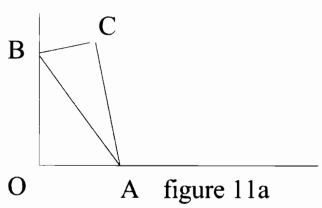


figure 11a

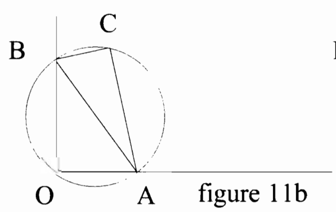


figure 11b

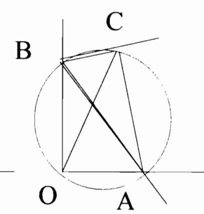


figure 11c

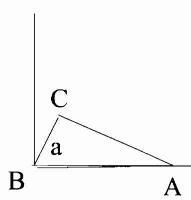


figure 11d

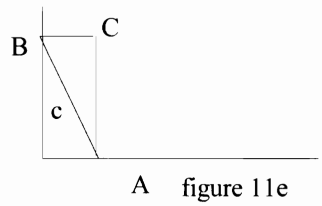


figure 11e

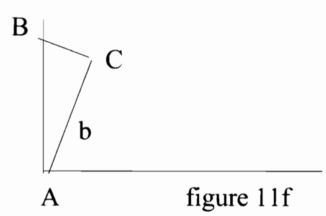


figure 11f

Pythagorean Triples

The Definition

A Pythagorean Triple (PT) is a set of three integers, $\{a, b, c\}$, in which $a^2 + b^2 = c^2$. Therefore, a and b are the lengths of the legs of a right triangle and c is the length of its hypotenuse. The ancient Babylonians were familiar with these triples for their famous cuneiform tablet, *Plimpton* 322 (c 2000 BC), lists fifteen Pythagorean triples. The author of this tablet apparently knew for any pair of integers, m and n (with $m > n$), the set $\{2mn, m^2 - n^2, m^2 + n^2\}$ always produced a PT. (If m and n are relatively prime these formulas produce a primitive PT, a PT with no common factors.)

The Problems

Part A – Generating Pythagorean Triples

Algorithm #1 For any pair of relatively prime integers, m and n (with $m > n$), $\{2mn, m^2 - n^2, m^2 + n^2\}$ is a PT.

1a. Use the above algorithm to generate ten primitive PTs.

1b. Prove $\{2mn, m^2 - n^2, m^2 + n^2\}$ is always a PT.

Algorithm #2 Take any two consecutive even or odd integers and add their reciprocals. The resulting numerator and denominator are a and b of a PT.

2a. Show this algorithm works for the numbers 13 and 15.

2b. Prove this algorithm always works.

Algorithm #3 Take any two fractions whose product is 2. Add 2 to each number and rewrite each as an improper fraction. Cross-multiply to produce two numbers a and b of a PT.

3a. Show this algorithm works for the numbers $\frac{3}{2}$ and $\frac{4}{3}$.

3b. Prove this algorithm always works.

Algorithm #4 Let a be any rational number greater than 1. Find two numbers, b and c, that differ by 1 and whose sum is a^2. The resulting set $\{a, b, c\}$ is a *rational* Pythagorean triple. Multiplying each member of the set by the common denominator of a, b, and c will produce a PT.

4a. Show this algorithm works for $\frac{7}{3}$.

4b. Prove this algorithm always works.

4c. The PT you found in part 'a' of this problem is the same PT which would have resulted if you had started with $\frac{5}{2}$! In addition, both $\frac{9}{5}$ and $\frac{7}{2}$ produce {28, 45, 53} and $\frac{4}{3}$ and 7 both produce {7, 24, 25}. When using this algorithm, for every fraction there exists a second fraction which produces the same PT. Looking at patterns in the examples given, what other fraction will produce the same results as $\frac{8}{5}$? Justify your claim.

4d. Using this algorithm, what rational number will produce the same PT as $\frac{m}{n}$? Prove your conjecture.

Algorithm #5 Consider the Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... Take any four consecutive Fibonacci numbers. The product of the outer two terms and twice the product of the inner two terms will be a and b of a PT. The value of c in the PT will also be a Fibonacci number and its subscript will be half the sum of the subscripts of the original four numbers! (Charles Raine)

5a. Show this algorithm works using the numbers 5, 8, 13, and 21.

5b. Prove this algorithm always works.

Part B – Properties of Pythagorean Triples

6. From your list of primitive PTs, notice that

a) either a or b is a multiple of 3.

b) either a or b is a multiple of 4.

c) either a, b, or c is a multiple of 5.

Prove each of these results.

7. Pierre Fermat proved every prime of the form $4n+1$ (that is, the primes 5, 13, 17, 29, 37, ...) is the sum of two squares in exactly one way, while a prime of the form $4n+3$ (such as 3, 7, 11, 19, 23, ...) is never the sum of two squares. Find two PTs that contain the prime number 929.

8. For any PT, $\{a, b, c\}$, the expressions, c^2+ab and c^2-ab, can both be written as the sum of two square numbers.

   a. Show this works for the PT $\{12, 35, 37\}$.

   b. Prove this always works.

9. For any PT $\{a, b, c\}$, it is always possible to find positive integers $r, s, and\ t$, such that $a=r+s$ and $b=s+t$ where $r^2+s^2+t^2$ is a square number.

   a. Show this works for the PT $\{8, 15, 17\}$.

   b. Prove it always is possible to find $r, s, and\ t$.

The Solutions

1\. a. Here are 18 primitive PTs:

4	3	5
12	5	13
8	15	17
24	7	25
20	21	29
40	9	41
12	35	37
60	11	61
28	45	53
56	33	65
84	13	85
16	63	65
48	55	73
80	39	89
112	15	113
36	77	85
72	65	97
144	17	145

1b. Show $(2mn)^2 + (m^2 - n^2)^2 = (m^2 + n^2)^2$

$$4m^2n^2 + m^4 - 2m^2n^2 + n^4 =$$
$$m^4 + 2m^2n^2 + n^4 =$$
$$(m^2 + n^2)^2 =$$

2a. $\frac{1}{13} + \frac{1}{15} = \frac{28}{195}$ and $28^2 + 195^2 = 197^2$.

2b. $\frac{1}{n} + \frac{1}{n+2} = \frac{2n+2}{n^2+2n}$

$$(2n+2)^2 + (n^2+2n)^2 =$$
$$4n^2 + 8n + 4 + n^4 + 4n^3 + 4n^2 =$$
$$n^4 + 4n^3 + 8n^2 + 8n + 4 = (n^2 + 2n + 2)^2$$

3a. $\frac{3}{2} \quad \frac{4}{3}$

$$\frac{3}{2} + 2 \quad \frac{4}{3} + 2$$
$$\frac{7}{2} \quad \frac{10}{3}$$
$$7 \times 3 = 21 \ldots\ldots 2 \times 10 = 20$$

$20^2 + 21^2 = 29^2$ and so $\{20, 21, 29\}$ is a PT.

3b. Take any fraction, $\frac{a}{b}$. Divide 2 by it to produce the second fraction $\frac{2b}{a}$. Adding 2 to each fraction will result in $\frac{a+2b}{b}$ and $\frac{2a+2b}{a}$. Cross-multiplying produces $a^2 + 2ab$ and $2ab + 2b^2$.

$$(a^2 + 2ab)^2 + (2ab + 2b^2)^2 = a^4 + 4a^3b + 8a^2b^2 + 8ab^3 + 4b^4 = (a^2 + 2ab + 2b^2)^2.$$

4a. $a = \frac{7}{3}$ $\quad b = \frac{1}{2}(\frac{49}{9}) - \frac{1}{2} = \frac{20}{9}$ $\quad c = \frac{1}{2}(\frac{49}{9}) + \frac{1}{2} = \frac{29}{9}$ $\quad 9 \cdot \{\frac{7}{3}, \frac{20}{9}, \frac{29}{9}\} = \{21, 20, 29\}$

4b. Let $a = \frac{m}{n}$. Then $c - b = 1$ and $c + b = \frac{m^2}{n^2}$.

Solving this system, $c = \frac{1}{2}(\frac{m^2}{n^2}) + \frac{1}{2} = \frac{m^2+n^2}{2n^2}$ and $b = \frac{1}{2}(\frac{m^2}{n^2}) - \frac{1}{2} = \frac{m^2-n^2}{2n^2}$.

Therefore, $2n^2 \cdot \{\frac{m}{n}, \frac{m^2-n^2}{2n^2}, \frac{m^2+n^2}{2n^2}\} = \{2mn, m^2-n^2, m^2+n^2\}$, a PT.

4c. Ans: $\frac{13}{3}$. If $\frac{m}{n}$ is one fraction, the the other is $\frac{m+n}{m-n}$.

4d. In 4b. above it was shown that $a = \frac{m}{n}$ produces the PT, $\{2mn, m^2-n^2, m^2+n^2\}$, so it must be shown that $a = \frac{m+n}{m-n}$ produces the same PT.

$$\{a,b,c\} = \{\frac{m+n}{m-n}, \frac{1}{2}(\frac{m+n}{m-n})^2 - \frac{1}{2}, \frac{1}{2}(\frac{m+n}{m-n})^2 + \frac{1}{2}\}$$

$$= \{\frac{m+n}{m-n}, \frac{(m+n)^2-(m-n)^2}{2(m-n)^2}, \frac{(m+n)^2+(m-n)^2}{2(m-n)^2}\}$$

$$= \{\frac{m+n}{m-n}, \frac{4mn}{2(m-n)^2}, \frac{2m^2+2n^2}{2(m-n)^2}\}$$

$$= \{\frac{m+n}{m-n}, \frac{2mn}{(m-n)^2}, \frac{m^2+n^2}{(m-n)^2}\}$$

$$= \{(m-n)(m+n), 2mn, m^2+n^2\}$$

$= \{m^2-n^2, 2mn, m^2+n^2\}$, the same PT (with a and b interchanged).

5a. $F_5 = 5$ $F_6 = 8$ $F_7 = 13$ $F_8 = 21$

$\frac{1}{2}(5+6+7+8) = 13$ $F_{13} = 233$

$F_5 \cdot F_8 = 5 \cdot 21 = 105$ $2 \cdot F_6 \cdot F_7 = 2 \cdot 8 \cdot 13 = 208$

$105^2 + 208^2 = 11025 + 43264 = 54289 = 233^2$

5b. $\frac{1}{2}(k + k+1 + k+2 + k+3) = 2k+3$. Therefore it must be shown that $\{F_k \cdot F_{k+3}, 2 \cdot F_{k+1} \cdot F_{k+2}, F_{2k+3}\}$ is a PT.

Let $F_k = m - n$ and $F_{k+1} = n$, then $F_{k+2} = m$ and $F_{k+3} = m + n$.

$F_k \cdot F_{k+3} = (m-n)(m+n) = m^2 - n^2$ and $2 \cdot F_{k+1} \cdot F_{k+2} = 2mn$.

Therefore, it must be shown that $F_{2k+3} = m^2 + n^2$.

Proof: $F_k = m - n$

$F_{k+1} = n$

$F_{k+2} = m$

$F_{k+3} = m + n$

$F_{k+4} = 2m + 1n = F_3 \cdot m + F_2 \cdot n$

$F_{k+5} = 3m + 2n = F_4 \cdot m + F_3 \cdot n$

$F_{k+6} = 5m + 3n = F_5 \cdot m + F_4 \cdot n \ldots$

...

$F_{k+k} = F_{2k} = F_{k-1} \cdot m + F_{k-2} \cdot n$

...

$F_{2k+3} = F_{k+2} \cdot m + F_{k+1} \cdot n$

$F_{2k+3} = m \cdot m + n \cdot n$

$F_{2k+3} = m^2 + n^2$

6a. Let $a = 2mn$ and $b = m^2 - n^2$. If m or n is a multiple of 3, then a is a multiple of 3. Otherwise, $m \equiv 1 \ mod\ 3$ or $m \equiv 2 \ mod\ 3$. In either case, $m^2 \equiv 1 \ mod\ 3$. The same is true for n, $n^2 \equiv 1 \ mod\ 3$. Therefore, $m^2 - n^2 \equiv 0 \ mod\ 3$ and b must be a multiple of 3.

6b. Again, let $a = 2mn$ and $b = m^2 - n^2$. If m or n is a multiple of 2, then a is a multiple of 4. Otherwise, $m \equiv 1 \ mod\ 4$ or $m \equiv 3 \ mod\ 4$. In either case, $m^2 \equiv 1 \ mod\ 4$. The same is true for n, $n^2 \equiv 1 \ mod\ 4$. Therefore, $m^2 - n^2 \equiv 0 \ mod\ 4$ and b must be multiple of 4.

6c. Again, let $a = 2mn$ and $b = m^2 - n^2$. If m or n is a multiple of 5, then a is a multiple if 5. Otherwise, $m \equiv 1 \ mod\ 5$, $m \equiv 2 \ mod\ 5$, $m \equiv 3 \ mod\ 5$, or $m \equiv 4 \ mod\ 5$. In the outer two cases, $m^2 \equiv 1 \ mod\ 5$ and, in the inner two cases, $m^2 \equiv 4 \ mod\ 5$. The same is true for n. There are four possible combinations of these cases: $m^2 \equiv 1 \ mod\ 5$ and $n^2 \equiv 1 \ mod\ 5$, $m^2 \equiv 1 \ mod\ 5$ and $n^2 \equiv 4 \ mod\ 5$, $m^2 \equiv 4 \ mod\ 5$ and $n^2 \equiv 1 \ mod\ 5$, or $m^2 \equiv 4 \ mod\ 5$ and $n^2 \equiv 4 \ mod\ 5$. In the outer two cases, $m^2 - n^2 \equiv 0 \ mod\ 5$ and so b would be a multiple of 5. In the inner two cases, $m^2 + n^2 \equiv 0 \ mod\ 5$ and so c would be a multiple of 5.

7. Since 929 is prime and equal to $4(232) + 1$, it must equal to $m^2 + n^2$. The easiest way to find the two squares that add up to 929 is to type $y_1 = \sqrt{}(929 - x^2)$ into your calculator and search the table for the integer solution. Since $929^2 = 23^2 + 20^2$, $m = 23$ and $n = 20$. So $a = 2(23)(20) = 920$ and $b = 23^2 - 20^2 = 129$. Therefore, $\{920, 129, 929\}$ is a PT. 929 can also equal $m^2 - n^2$. Since $m^2 - n^2 = (m - n)(m + n)$ and 929 is prime, $m - n = 1$ and $m + n = 929$, resulting in $m = 465$ and $n = 464$, producing the PT $\{929, 431520, 431521\}$.

8a. $37^2 + (12)(35) = 1789 = 5^2 + 42^2$. (1789 is a (4k + 1) - type prime!)

$37^2 - (12)(35) = 949 = 7^2 + 30^2$. (Also, $18^2 + 25^2$ works.)

8b. Notice that $12 + 35 + 37 = 84$, $12 - 35 + 37 = 14$, $-12 + 35 + 37 = 60$, and $12 + 35 - 37 = 10$. (These patterns are even more apparent if the theorem is applied to more PTs.) Therefore, it must be shown that

$$c^2 + ab = \left(\frac{a+b+c}{2}\right)^2 + \left(\frac{a+b-c}{2}\right)^2 = \frac{(a+b)^2 + 2c(a+b) + c^2}{4} + \frac{(a+b)^2 - 2c(a+b) + c^2}{4}$$

$$= \frac{2a^2 + 4ab + 2b^2 + 2c^2}{4} = \frac{4ab + 4c^2}{4} = ab + c^2$$

$$c^2 - ab = \left(\frac{a-b+c}{2}\right)^2 + \left(\frac{-a+b+c}{2}\right)^2 = \frac{(a-b)^2 + 2c(a-b) + c^2}{4} + \frac{(-a+b)^2 + 2c(-a+b) + c^2}{4}$$

$$= \frac{2a^2 - 4ab + 2b^2 + 2c^2}{4} = \frac{4ab - 4c^2}{4} = ab - c^2$$

9a.

$8 = r + s$	$15 = s + t$	$r^2 + s^2 + t^2$
$= 1 + 7$	$= 7 + 8$	$1^2 + 7^2 + 8^2 = 114$
$= 2 + 6$	$= 6 + 9$	$2^2 + 6^2 + 9^2 = 121$ **
$= 3 + 5$	$= 5 + 10$	$3^2 + 5^2 + 10^2 = 134$
$= 4 + 4$	$= 4 + 11$	$4^2 + 4^2 + 11^2 = 153$
$= 5 + 3$	$= 3 + 12$	$5^2 + 3^2 + 12^2 = 178$
$= 6 + 2$	$= 2 + 13$	$6^2 + 2^2 + 13^2 = 209$
$= 7 + 1$	$= 1 + 14$	$7^2 + 1^2 + 14^2 = 246$

Therefore, $r = 2, s = 6, \text{and } t = 9$.

9b. Looking at a few more examples:

$\{8,15,17\} \rightarrow r = 2,\ s = 6,\ t = 9$

$\{3,4,5\} \rightarrow r = 1,\ s = 2,\ t = 2$

$\{5,12,13\} \rightarrow r = 1,\ s = 4,\ t = 8$

$\{7,24,25\} \rightarrow r = 1,\ s = 6,\ t = 18$

Therefore, it seems $r = c - b$ *and* $t = c - a$ and since $a = r + s$, $s = a + b - c$.

Hence, it must be shown that for any PT $\{a, b, c\}$,

$(c - b)^2 + (a + b - c)^2 + (c - a)^2$ is a square number.

$c^2 - 2bc + b^2 + (a+b)^2 - 2c(a+b) + c^2 + c^2 - 2ac + a^2$

$3c^2 - 4bc - 4ac + 2a^2 + 2ab + 2b^2$

For this to be a perfect square, the coefficient of c^2 must be a square. Since $c^2 = a^2 + b^2$, the expression can be rewritten as: $4c^2 - 4c(b+a) + a^2 + 2ab + b^2 = 4c^2 - 4c(a+b) + (a+b)^2 = (2c-(a+b))^2$.

The Extensions

1. In a PT, is either $a, b, a-b, or\ a+b$ always a multiple of 7.

2. Is c always of the form $4n+1$?

3. What numbers never occur in a primitive PT?

4. In an *ordered primitive* PT $\{a, b, c\}$ with $a < b < c$, is c always equal to $b+1$, if a is odd, and equal to $b+2$, if a is even?

5. Determine all triples $\{a, b, c\}$ such that $3^a + 4^b = 5^c$.

6. Determine all triples $\{a, b, c\}$ such that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

7. How do you generate all the Pythagorean Quadruples i.e. $\{a, b, c, d\}$ such that $a^2 + b^2 + c^2 = d^2$?

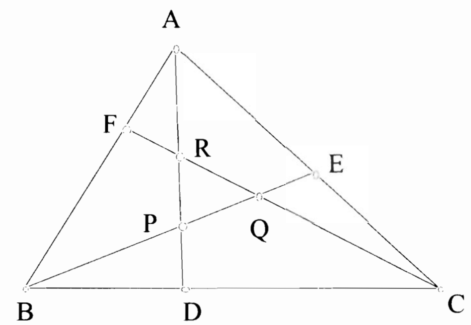
8. Define D, the distance between two PTs, equal to $|c_1 - c_2|$ and H, the height of a PT, equal to $c - b$. Let $\{a_0, b_0, c_0\}$ be a PT of height H, D be a positive integer, and $\beta = \frac{D}{H}$. If βa_0 and $\frac{\beta D}{2}$ are integers, then $a_{k+1} = a_k + D, \quad b_{k+1} = \beta a_k + b_k + \frac{\beta D}{2}, \quad c_{k+1} = \beta a_k + c_k + \frac{\beta D}{2}$ are recursive formulas that produce **all** PTs of height H and the distance between each pair is precisely D. In the case of H =1, $\{a_0, b_0, c_0\} = \{3, 4, 5\}$.

More about this theorem can be found in The College Mathematics Journal, Vol. 30, No. 2, March 2000..

Cevians

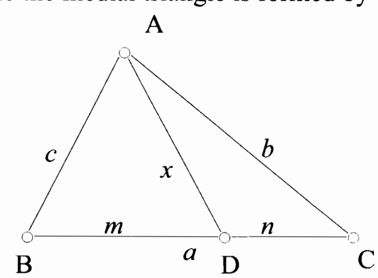
The Definitions and Theorems

A cevian is a segment which joins the vertex of a triangle with a point on its opposite side (or its extension.) Throughout this problem set, unless it is stated to the contrary, the three cevians of ΔABC will be labeled $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.



Some cevians have special names like altitudes, medians, and angle bisectors. Sometimes, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent at point P, while other times they form an interior cevian triangle, called ΔPQR. When concurrent, oftentimes, point P has a special name like orthocenter, centroid, incenter, or the Nagel, Gergonne, Lemoine, or Miquel point.

Triangle ΔDEF is called the inscribed cevian triangle. The orthic and medial triangles are special inscribed cevian triangles. The orthic triangle is formed when the cevians are altitudes while the medial triangle is formed by medians.



The following three theorems may be useful in solving this problem set:

Stewart's Formula: The length of a cevian, x, and the sides of a triangle are related by the following formula: $c^2n + b^2m = x^2a + mna$

Ceva's Theorem: The three cevians, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, are concurrent if and only if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$

Routh's Theorem: If K_{ABC} is the area of ΔABC, then the area of the interior cevian triangle PQR is

$$K_{PQR} = \frac{(rst - 1)^2}{(st + s + 1)(tr + t + 1)(rs + r + 1)} \cdot K_{ABC}$$ and the area of the inscribed cevian triangle DEF is

$$K_{DEF} = \frac{rst}{(s + 1)(t + 1)(r + 1)} \cdot K_{ABC}, \text{ where } \frac{BD}{DC} = r,\ \frac{CE}{EA} = s,\ \frac{AF}{FB} = t.$$

The Problems

Part A – Cevian Length and Concurrency

1. Show that the length of the median from vertex C of ΔABC, $x = \frac{\sqrt{2a^2 + 2b^2 - c^2}}{2}$.

2. Show that the length of the altitude from vertex C of ΔABC,
$$x = \frac{\sqrt{(a + b - c)(b + c - a)(c + a - b)(a + b + c)}}{2c}.$$

3. Show that the length of the angle bisector from vertex C of ΔABC,
$$x = \frac{\sqrt{a \cdot b \cdot (a + b - c)(a + b + c)}}{a + b}.$$

4. If the altitude, median and angle bisector from vertex C of a ΔABC divide $\angle ACB$ into four equal angles. Find the degree measures of $\angle A$, $\angle B$, and $\angle ACB$.

5. In ΔABC, the median $\overline{AD}$, the altitude $\overline{BE}$, and the angle bisector $\overline{CF}$ are concurrent. Show that.
$$\frac{a + b}{a - b} = \frac{b^2}{a^2 - c^2}$$

6. In ΔABC, point D is on $\overline{BC}$ and half way around the triangle from vertex A, i.e., $AC + CD = AB + BD$, and E is on $\overline{AC}$ and half way around the triangle from vertex B, and F is on $\overline{AB}$ and half way around the triangle from C. Prove that the cevians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent.

7. Prove that the orthocenter of ΔABC is the incenter of its orthic triangle ΔDEF.

Part B – Areas of Cevian Triangles

8. The sides of ΔABC have lengths $14, 25,$ and $\sqrt{149}$. The three cevians divide each side in the ratio 3:5 and form an interior cevian triangle ΔPQR. Find the area of ΔPQR.

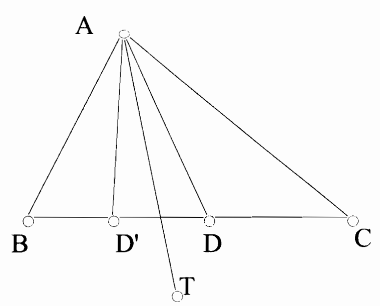
9. The cevians of a triangle partition the sides into segments whose ratios are 1:1, 4:1, and 7:1, consecutively. If the area of the inscribed cevian triangle is 28, what is the area of the interior cevian triangle?

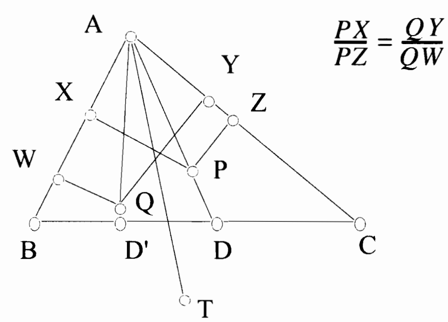
10. The cevians of a triangle partition the sides into segments whose ratios are each $1{:}n$. Find a <u>simplified</u> rational function which expresses $\frac{K_{PQR}}{K_{ABC}}$ in terms of n.

Part C – Symmedians

If two cevians from the same vertex of a triangle are symmetrical with respect to the angle bisector of that angle, i.e. $\angle D'AT \cong \angle DAT$ in figure below, they are said to be isogonal and form a pair of <u>isogonal conjugates</u>.

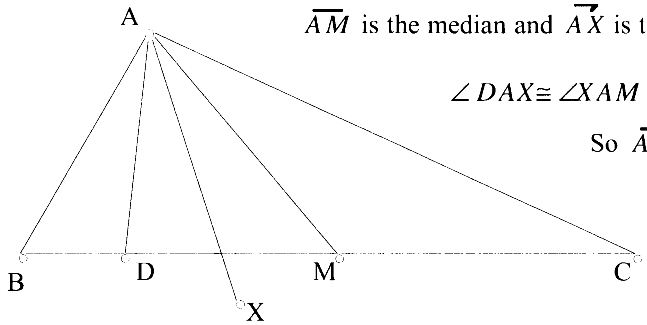
<u>The Isogonal Theorem</u>: If $\overline{AD}$ and $\overline{AD'}$ are cevians with point P on $\overline{AD}$ and point Q on $\overline{AD'}$, then $\overline{AD}$ and $\overline{AD'}$ are isogonal if and only if the ratio of distances from P to $\overline{AB}$ and $\overline{AC}$ are inversely proportional to the ratio of the distances from point Q to $\overline{AB}$ and $\overline{AC}$. ($\overrightarrow{AT}$ bisects angle BAC.)





11. Prove, if the cevians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent, then so are their isogonal conjugates, $\overline{AD'}$, $\overline{BE'}$, and $\overline{CF'}$.

A <u>symmedian</u> is a cevian which is isogonal to a median. Since medians are concurrent at the centroid, from problem 1 above, the symmedians of a triangle must also be concurrent. (Their point of concurrency is called the Lemoine point of the triangle.)



$\overline{AM}$ is the median and $\overrightarrow{AX}$ is the angle bisector.

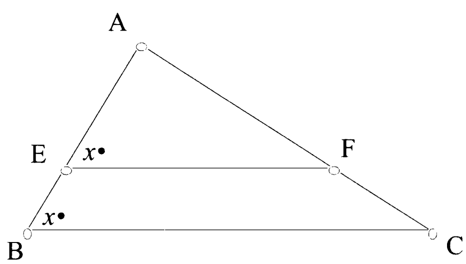
$\angle DAX \cong \angle XAM$

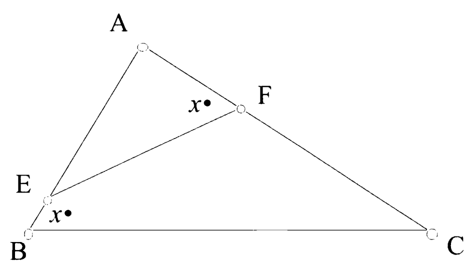
So $\overline{AD}$ is a symmedian !

12. Prove P is a point on symmedian $\overline{AD}$ if and only if the ratio of the distances from P to $\overline{AB}$ and $\overline{AC}$ is proportional to $\frac{AB}{AC}$.

13. Prove if $\overline{AD}$ is a symmedian, then $\frac{BD}{DC} = \frac{AB^2}{AC^2}$.

In ΔABC let E be a point on $\overline{AB}$ and F a point on $\overline{AC}$. If $\Delta AEF \approx \Delta ABC$ then $\overline{EF}$ is a <u>parallel</u> of side $\overline{BC}$ of ΔABC and if $\Delta AFE \approx \Delta ABC$ then $\overline{EF}$ is an <u>antiparallel</u> of side $\overline{BC}$ of ΔABC.





14. In ΔABC we know the median $\overline{AM}$ bisects every parallel of side $\overline{BC}$. Prove the symmedian $\overline{AD}$ bisects every antiparallel of side $\overline{BC}$.

The Solutions

1. Using Stewart's Formula, $a^2m + b^2n = x^2c + mnc$, let $m = n = \frac{c}{2}$. Then $a^2\frac{c}{2} + b^2\frac{c}{2} = x^2c + \frac{c}{2}\frac{c}{2}c \Rightarrow$

$$\frac{a^2}{2} + \frac{b^2}{2} = x^2 + \frac{c^2}{4} \Rightarrow \frac{2a^2 + 2b^2 - c^2}{4} = x^2 \Rightarrow \frac{\sqrt{2a^2 + 2b^2 - c^2}}{2} = x$$

2. Area of the triangle, $K = \frac{1}{2}cx \Rightarrow x = \frac{2K}{c}$.

Area of the triangle, $K = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$

$$K = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2} - a\right)\left(\frac{a+b+c}{2} - b\right)\left(\frac{a+b+c}{2} - c\right)}$$

$$K = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)}$$

$$K = \frac{\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}}{4}$$

$$x = \left(\frac{2}{c}\right)\frac{\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}}{4}$$

$$x = \frac{\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}}{2c}.$$

3. By the Angle Bisector Theorem, $\frac{m}{n} = \frac{b}{a}$ and $m + n = c$. Therefore, $m = \frac{bc}{a+b}$ and $n = \frac{ac}{a+b}$. Plugging these into Stewart's Formula yields: $a^2\frac{bc}{a+b} + b^2\frac{ac}{a+b} = x^2c + \frac{bc}{a+b}\frac{ac}{a+b}c \Rightarrow \frac{abc(a+b)}{a+b} = x^2c + \frac{abc^3}{(a+b)^2} \Rightarrow$

$$\Rightarrow ab - \frac{abc^2}{(a+b)^2} = x^2 \Rightarrow \frac{ab(a+b)^2 - abc^2}{(a+b)^2} = x^2 \Rightarrow \frac{ab((a+b)^2 - c^2)}{(a+b)^2} = x^2 \Rightarrow \frac{\sqrt{ab(a+b-c)(a+b+c)}}{a+b} = x.$$

4. $AM = MB$ and so $AH + HM = HB - HM$.
Therefore,
$h\tan x + h\tan 2x = h\tan 3x - h\tan 2x$
$\tan x + 2\tan 2x = \tan(x + 2x)$
$\tan x + 2\tan 2x = \frac{\tan x + \tan 2x}{1 - \tan x \tan 2x}$

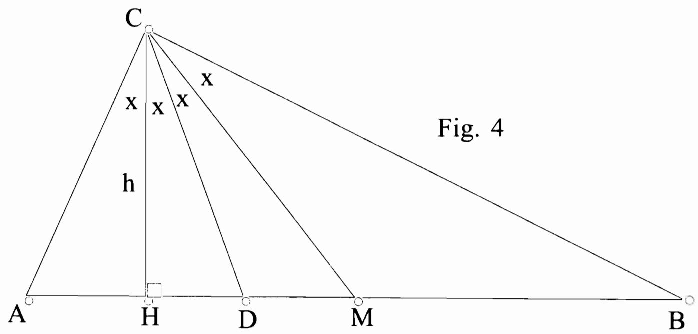


Fig. 4

$\tan x - \tan^2 x \tan 2x + 2\tan 2x - 2\tan x \tan^2 2x = \tan x + \tan 2x \Rightarrow$

$-\tan^2 x \tan 2x + \tan 2x - 2\tan x \tan^2 2x = 0 \Rightarrow \quad -\tan^2 x + 1 - 2\tan x \tan 2x = 0 \Rightarrow$

$-\tan^2 x + 1 - \dfrac{4\tan^2 x}{1-\tan^2 x} = 0 \Rightarrow \quad \tan^4 x - 6\tan^2 x + 1 = 0 \quad \Rightarrow \quad \tan x = \sqrt{3 \pm 2\sqrt{2}} \quad \Rightarrow \quad x = \tan^{-1}\sqrt{3 \pm 2\sqrt{2}} \quad \Rightarrow$

$x = 22.5°$ *or* $67.5° \quad \Rightarrow \quad x = 22.5°$. Therefore, $\angle A = 67.5°$, $\angle B = 22.5°$, and $\angle C = 90°$.

5. By Ceva's Theorem, $\frac{AD}{DB} \cdot \frac{BM}{MC} \cdot \frac{CH}{HA} = 1$. But since M is a midpoint, $\frac{BM}{MC} = 1$ and since $\overline{CD}$ is an angle bisector, $\frac{AD}{DB} = \frac{b}{a}$. Therefore, $\frac{CH}{HA} = \frac{a}{b}$. Let $CH = at$ and $HA = bt$, then $at + bt = b$, or $t = \frac{b}{a+b}$.

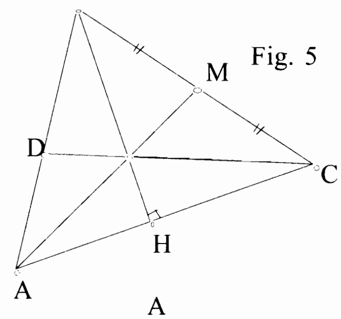


Fig. 5

By the Pythagorean Theorem, $BH^2 = AB^2 - HA^2$, so $BH^2 = c^2 - b^2t^2$ and $BH^2 = BC^2 - CH^2$, so $BH^2 = a^2 - a^2t^2$.

Therefore, $c^2 - b^2t^2 = a^2 - a^2t^2$ and $t^2 = \frac{a^2 - c^2}{a^2 - b^2}$. Substituting for t, $\left(\frac{b}{a+b}\right)^2 = \frac{a^2-c^2}{a^2-b^2}$ and so $\frac{b^2}{a^2-c^2} = \frac{a+b}{a-b}$.

6. Let s = the semi perimeter of ΔABC, then AE = $s - c$, EC = $s - a$, CD = $s - b$, DB = $s - c$, BF = $s - a$, and FA = $s - b$.

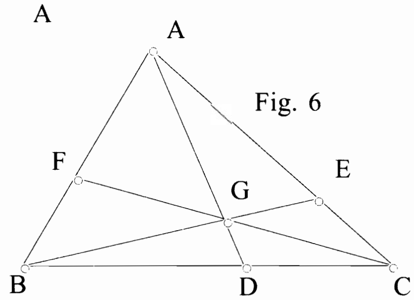


Fig. 6

$$\frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BF}{FA} = \frac{s-c}{s-a} \cdot \frac{s-b}{s-c} \cdot \frac{s-a}{s-b} = 1.$$

Therefore, the cevians are concurrent. This is known as **Geronne's Point.**

7. ΔDEF is the orthic triangle of ΔABC, with P the ortho center, the intersection of the altitudes. $\Delta AFP \approx \Delta CDP$ and so $\angle FAP \cong \angle DCP$. Since quadrilateral $AEPF$ has two right angles it must be cyclic. The same is true for quadrilateral $CDPE$.

Because they intercept the same arc, $\angle FAP \cong \angle FEP$ and $\angle DCP \cong \angle PED$. Therefore, $\angle FEP \cong \angle PED$.

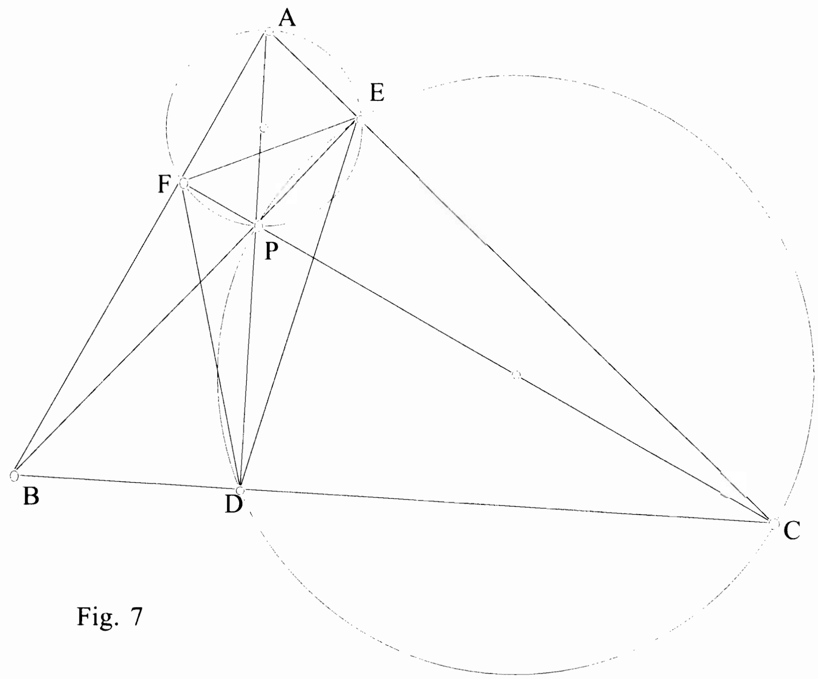


Fig. 7

The same reasoning can be used to show $\angle EDP \cong \angle FDP$ and $\angle DFP \cong \angle EFP$ and so $\overrightarrow{DA}$, $\overrightarrow{EB}$, and $\overrightarrow{FC}$ are bisectors of the angles of ΔDEF and so P is the incenter of ΔDEF.

8. By Hero's Formula the area of ΔABC is $K_{ABC} = \sqrt{s(s-14)(s-25)(s-\sqrt{149})}$ where $s = \frac{\sqrt{149}+39}{2}$.

$$= \sqrt{\left(\frac{\sqrt{149}+39}{2}\right)\left(\frac{\sqrt{149}+11}{2}\right)\left(\frac{\sqrt{149}-11}{2}\right)\left(\frac{39-\sqrt{149}}{2}\right)} = \frac{\sqrt{(39^2-149)(149-11^2)}}{4} = \frac{\sqrt{28456}}{4} = 49.$$ Using

Routh's Theorem, the area of ΔPQR is $K_{PQR} = \dfrac{\left(\frac{3}{5}-1\right)^2}{\left(\frac{3}{5}\right)^2+\frac{3}{5}+1} \cdot K_{ABC} = \left(\frac{4}{49}\right)(49) = 4$.

9. Using Routh's Theorem for the inscribed cevian triangle,

$$28 = \frac{(1)(4)(7)}{(2)(5)(8)} \cdot \Delta ABC \Rightarrow 28 = \frac{28}{80} \cdot \Delta ABC \Rightarrow 80 = \Delta ABC$$

Let K = the area of the interior cevian triangle, $K = \dfrac{(28-1)^2}{(28+4+1)(7+7+1)(4+1+1)} \cdot 80 = \dfrac{216}{11}$.

10.
$$K_{PQR} = \frac{(n^3-1)^2}{(n^2+n+1)(n^2+n+1)(n^2+n+1)} \cdot K_{ABC}$$
$$\frac{K_{PQR}}{K_{ABC}} = \frac{\left((n-1)(n^2+n+1)\right)^2}{(n^2+n+1)^3}$$
$$\frac{K_{PQR}}{K_{ABC}} = \frac{(n-1)^2}{(n^2+n+1)} = \frac{(n^2-2n+1)}{(n^2+n+1)}$$

11. Let cevians $\overline{AD}$, $\overline{BE}$, and $\overline{CE}$ be concurrent at point P and let Q be the intersection $\overline{AD'}$ *and* $\overline{BE'}$, the isogonal conjugates of $\overline{AD}$ and $\overline{BE}$. Let the distances from P to sides $\overline{BC}$, $\overline{AC}$, and $\overline{BC}$ be a', b', and c', respectively, and the distances from Q to the sides $\overline{BC}$, $\overline{AC}$, and $\overline{BC}$ be a'', b'', and c'', respectively. Using $\overline{AD}$ *and* $\overline{AD'}$ and The Isogonal Theorem, $\frac{b'}{c'} = \frac{c''}{b''}$. Using $\overline{BE}$ and $\overline{BE'}$ and The Isogonal Theorem, $\frac{c'}{a'} = \frac{a''}{c''}$. Combining the two equations using multiplication yields
$$\frac{b'}{c'} \cdot \frac{c'}{a'} = \frac{c''}{b''} \cdot \frac{a''}{c''}$$
$$\frac{b'}{a'} = \frac{a''}{b''}.$$

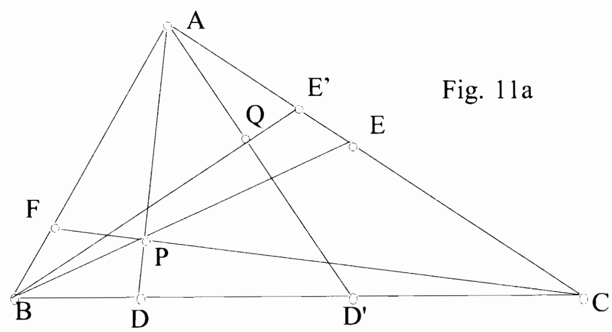


Fig. 11a

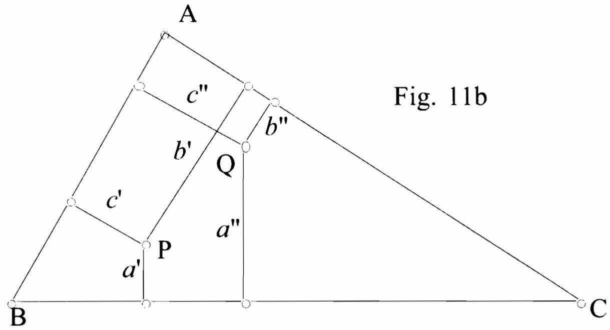


Fig. 11b

Let $\overline{CF'}$ be a cevian from C through point Q. By The Isogonal Theorem, $\overline{CF}$ *and* $\overline{CF'}$ must be isogonal. Therefore, isogonals of concurrent cevians are concurrent.

12. $\Rightarrow$ Assume P is a point on symmedian $\overline{AD}$. By the Isogonal Theorem, $\frac{x}{y} = \frac{u}{t}$. Because M is a midpoint, $K_{ABM} = K_{ACM}$ and $\frac{1}{2}ct = \frac{1}{2}bu$. Therefore. $\frac{c}{b} = \frac{u}{t}$ and so $\frac{x}{y} = \frac{c}{b}$.

$\Leftarrow$ Assume $\frac{x}{y} = \frac{c}{b}$ and M is the midpoint of $\overline{BC}$. Using the same area logic as above, $\frac{c}{b} = \frac{u}{t}$ and

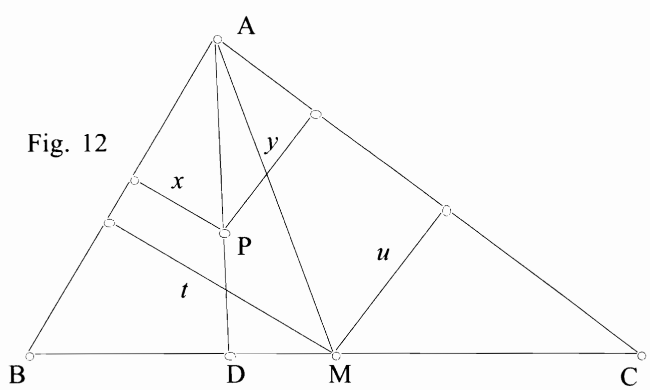


Fig. 12

so, by The Isogonal Theorem, $\overline{AP}$ ($\overline{AD}$) and $\overline{AM}$ are isogonal. Since $\overline{AM}$ is a median, $\overline{AD}$ must be a symmedian.

13. Let $\overline{AD}$ be a symmedian and let $BD = m$ and $DC = n$. From problem 12 above, $\frac{x}{y} = \frac{c}{b}$.

Because they share the same altitude, $\frac{K_{ABD}}{K_{ADC}} = \frac{m}{n}$.

But $K_{ABD} = \frac{1}{2}xc$ and $K_{ADC} = \frac{1}{2}yb$. So

$$\frac{m}{n} = \frac{xc}{yb} = \left(\frac{x}{y}\right)\left(\frac{c}{b}\right) = \left(\frac{c}{b}\right)\left(\frac{c}{b}\right) = \frac{c^2}{b^2}.$$

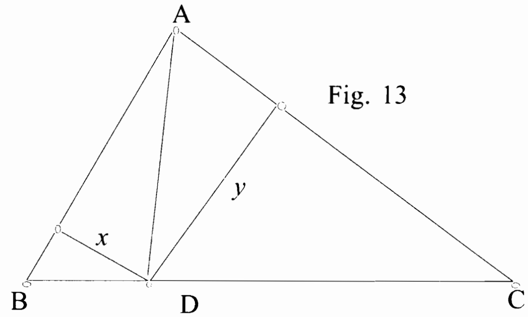


Fig. 13

14. Consider ΔABC with $\overline{DE}$, a parallel to side $\overline{BC}$ and $\overrightarrow{AX}$, the angle bisector (Figure 14a). Let $\overline{D'E'}$ be the reflection of $\overline{DE}$ through $\overrightarrow{AX}$. Using congruent triangles, it is easy to prove that $\Delta ADE \cong \Delta AD'E'$ and so $\angle ADE \cong \angle AD'E'$. Therefore, $\overline{D'E'}$ is an antiparallel to side $\overline{BC}$.

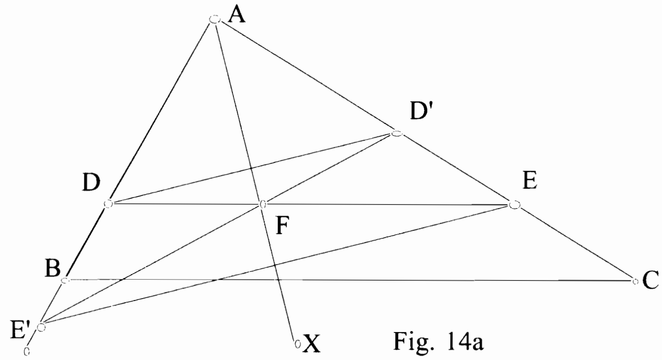


Fig. 14a

Using Figure 14b, since the median $\overline{AM}$ bisects $\overline{DE}$ at G and the fact that the reflection transformation preserves distances, its reflection through $\overrightarrow{AX}$ will bisect the reflection of $\overline{DE}$ through $\overrightarrow{AX}$. Therefore, the symmedian from A bisects any antiparallel to $\overline{BC}$.

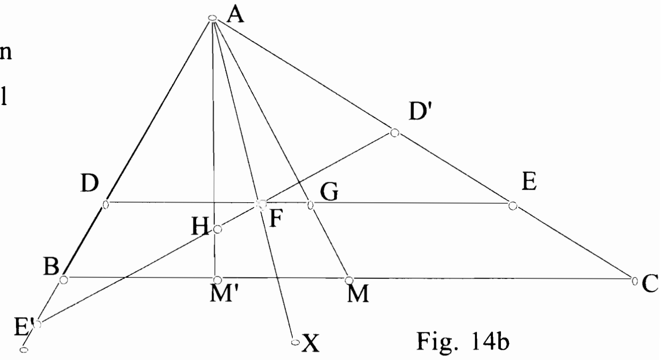


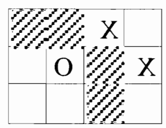
Fig. 14b

Insane Tic-Tac-Toe

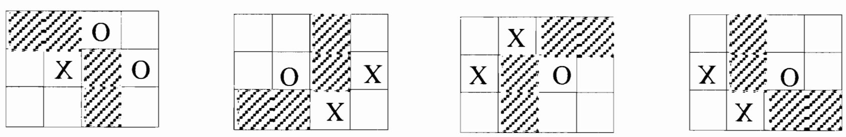
The Definitions

In the game of Tic-Tac-Toe (also called Naughts and Crosses) two players, one using X's and the other using O's, take turns placing their X's or O's in a 3 by 3 grid. The winner of the game is the first player to get three X's or three O's in a row horizontally, vertically, or diagonally. The game of Insane Tic-Tac-Toe (IT^3) is played on a 3 by 4 grid, but **on any turn a player may place either an X or an O in the grid**. Again the winner is the first player to complete three X's or three O's in a row horizontally, vertically, or diagonally.

Consider the following game board (a partially played game):



It can be inverted (by interchanging all X's and O's), reflected through a horizontal line, reflected through a vertical line, or rotated 180° to become the following boards.



Since the strategy for winning on any of these boards is equivalent, we will say these four boards are equivalent. Therefore, any two boards are equivalent if one can be changed into the other by any combination of inversion, horizontal or vertical reflection, or rotation.

A death cell on the IT^3 board is a cell in which putting any mark, either an X or an O, will allow the opponent to win on his or her next move. The death cells have been shaded in the game boards above.

In analyzing the game of IT^3, a round will be considered a move (or turn) by player A followed by a move (or turn) by player B. We will assume in all problems that each player is trying to win and playing smart. A person playing smart will not provide the opponent a winning opportunity (2 of 3 X's or O's in a row) unless forced to do so. Because of the play smart rule, the following boards will never happen:

	O	O	
X			
	X		

		X	
X			
	O		O

	O		
X			
	X		O

(It would be impossible for two O's to be in a row as on these boards, if the players are playing smart.)

ARML Power Contest – February 2002 – Insane Tic-Tac-Toe

The Problems

In the following problems you can refer the a game board using a 3 by 4 grid and/or refer to the individual cells using the following format: Cutting and taping grids from the grid sheets provided may make your work more readable.

1	2	3	4
5	6	7	8
9	10	11	12

Set A

1. Determine all the boards that are equivalent to each of the following game boards

a.

		X	
	O		

b.

X			
	O	X	
			O

2. After the first round, if the players play smart, how many nonequivalent game boards are possible? Justify your answer.
3. Determine five nonequivalent game boards that have seven death cells. Be sure to shade in the seven death cells.

Set B

When Ann told her sister Beth about Insane Tic-Tac-Toe. Beth was a little suspicious, especially when Ann said, "You can always go first!" Beth said, "I'll play, but you must go first." After a few games Beth realized a simple non losing strategy. Every time it was her turn, unless she could win she would simply do the opposite of what Ann did--in other words, rotate and invert Ann's last move. This is an example of how the first rounds might go:

		X	

Ann

	O	X	

Beth

Round 1

X			
	O	X	

Ann

X			
	O	X	
			O

Beth

Round 2

Throughout the problems in set B, Beth will use this copy-cat strategy.

4. After the first round, if both players are playing smart, how many nonequivalent game boards are there? Justify your answer. Make a representation for each game board and shade in all death cells.

5. After the second round, if Beth has not already won and both players are playing smart, there are eight nonequivalent game boards. Prove this and make a representation for each game board and shade in all death cells.
6. For each of the game boards in problem 5 above determine how many rounds Ann will last if Beth continues this strategy.

ARML Power Contest – February 2002 – Insane Tic-Tac-Toe

Set C

After awhile Ann complained because she was never winning. So in an effort to make the game more fair, they decided that Ann would be allowed to place two X's or two O's on the first turn. Beth's original strategy no longer worked. Soon Ann was winning every game!

Throughout the problems in set C, Ann will make two X's on her first turn.

7. How many nonequivalent opening moves does Ann have, if Ann plays smart? Justify your answer.

8. If both players have played smart, how many nonequivalent game boards are there after the first round? Justify your answer. Be careful of double counting!

			X
	X		

Ann determined that if she always started with X's on cells 4 and 6, she could always win! Assume that she always used this opening move answer the following questions to prove she can always win regardless of how smart Beth plays. In each proof, you may stop when all the remaining cells are death cells. Careful!! After some moves, some death cells come back to life!

9a. If Ann always started with the above opening move and both players played smart, how many nonequivalent game boards are there after the first round? Justify your answer.

9b. If Beth's first move was an O in cell 1, then Ann's second move was an O in cell 11. (Likewise, if Beth's first move was an O in cell 11, Ann's second move was an O in cell 1.) Prove that Ann can always win when starting off this way.

O			X
	X		
		O	

9c. If Beth's first move was an O in cell 3, then Ann's second move was an O in cell 9. (Likewise, if Beth's first move was an O in cell 9, Ann's second move was an O in cell 3.) Prove that Ann can always win when starting off this way.

		O	X
	X		
O			

9d. If Beth's first move was an O in cell 2, then Ann's second move was an O in cell 10. (Likewise, if Beth's first move was an O in cell 10, Ann's second move was an O in cell 2.) Prove that Ann can always win when starting off this way.

	O		X
	X		
	O		

9e. If Beth's first move was an O in cell 5, then Ann's second move was an O in cell 7. (Likewise, if Beth's first move was an O in cell 7, Ann's second move was an O in cell 5.) Prove that Ann can always win when starting off this way.

			X
O	X	O	

9f. If Beth's first move was an O in cell 12, what must Ann's second move be to guarantee a win? Show this move guarantees a win regardless of where Beth moves and that Beth could win if Ann chooses any other second move.

			X
	X		
			O

9g. If Beth's first move was an O in cell 8 and Ann's second move is in cell 12, a win can be guaranteed for either Ann or Beth depending on whether Ann plays an X or an O. Show this is true.

			X
	X		O

<u>Set D</u>

Now on the losing end of the stick every time, Beth is no longer willing to give Ann two X's or O's on the first move. Instead they return to the original rules but draw the game board on a torus so that the top of the grid is connected to the bottom of the grid and the right side of the grid is connected to the left side. Now there are no edges and every cell is adjacent to eight cells.

10. If Ann opens with an X in cell 1, prove Beth must play an O.

11. Prove Beth will always win on a torus!

The Solutions

1a. There are seven:

	X		
		O	

	O		
		X	

		O	
	X		

		O	
	X		

	O		
		X	

	X		
		O	

		X	
	O		

1b. There are three:

O			
	X	O	
			X

			O
	O	X	
X			

			X
	X	O	
O			

2. After the first round the game board must contain two different marks or two of the same marks.

Case 1-Two different marks. The X can be in one of four cells: a corner cell, a long side cell, a short side cell, or a center cell.

X	O	O	O
O	O	O	O
O	O	O	O

If placed in a corner cell, there are eleven possible locations for the O. [11]

If placed in a long side cell, there are eleven possible locations for the O but four of these are inversions of a board already counted. [7]

O	X	O	O
O	O	O	O
O	O	O	O

If placed in a short side cell, there are eleven possible locations for the O but eight are inversions of boards already counted. [3]

O	O	O	O
X	O	O	O
O	O	O	O

If placed in a center cell, there are eleven possible locations for the O but all but one are inversions of boards already counted. [1]

O	O	O	O
O	X	O	O
O	O	O	O

Case 2 - Two marks the same. Again the first X can be placed in one of four cells:

X			X
		X	
	X		X

If placed in a corner cell, there are only five locations for the second X. Remember they play smart! [5]

	X		
X			X
		X	

If placed in a long side cell, there are four possible locations for the second X but one of these is an inversion of a board already counted. [3]

X			X

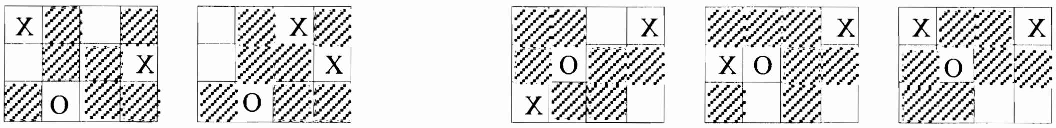
If placed in a short side cell, there are seven possible locations for the second X but all but one of these is an inversion of a board already counted. [1]

	X		

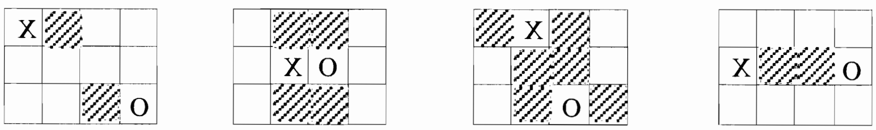
If placed in a center cell, there are only two possible locations for the second X but both of these are inversions of a boards already counted. [0]

Therefore, there are 31 nonequivalent game boards after the first round.

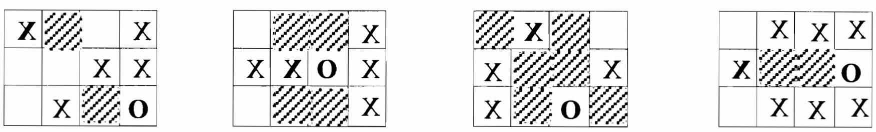
3. Obvious the board must contain an X and an O for there to be any death cells. Furthermore, it is easy to see that the board must contain a second X or O. Consider a board with one O and two X's. There are only four possible locations for the O: a corner cell, a long side cell, a short side cell, or a center cell. If O is in a corner cell, there are only six possible death cells and, if O is place in a short side cell, there are only four possible death cells. If O is in a long side cell, there are eight possible death cells and, if O is place in a center cell, there are nine possible death cells. Therefore, the O must be in a long side or center cell. If O is in a long side cell, there are only two arrangements of the X's, while if O is in a center cell, there are three arrangements for the X's



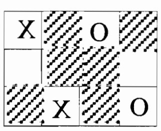
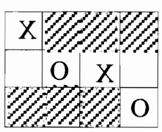
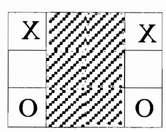
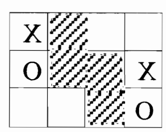
4. Again the X can be in one of four cells: a corner cell, a long side cell, a short side cell, or a center cell. The O must be in the "opposite" corner cell, long side cell, short side cell, or center cell.



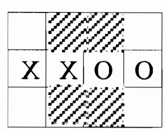
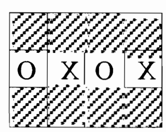
5. Again because of the symmetry of Beth's second move and the fact that it is the inversion of Ann's second move, we only need to worry about where the next X goes. In each of the diagrams below, the light X's represent the possible locations for the second X for each of the first round game boards.



The first board produces these four nonequivalent game boards:

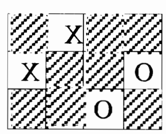
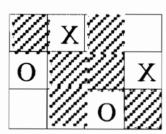
a.  b.  c.  d. 

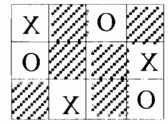
The second board produces only two new nonequivalent game boards (e and f):

e.  f. 

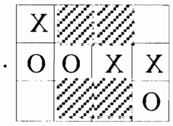
And the third produces two new nonequivalent game boards (g and h):

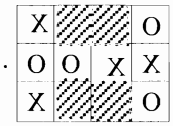
While the fourth produces no new nonequivalent game boards.

g.  h. 

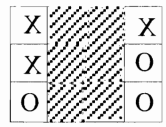
a.


6a. Game board (a) at the end of round three is at the right: Because only death cells remain, Beth will win on the next (fourth) round.

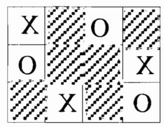
b1.


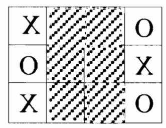
b2.


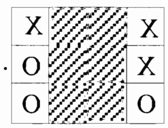
6b. Game board (b1) at the end of round three is to the right: The board after round four (b2) is to its right. Because only death cells remain, Beth will win in round five.

c.


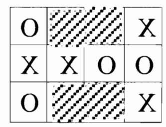
6c. The two possible third rounds of board (c) are vertical reflections. Beth will win on the next (fourth) round.

d1.


d2.


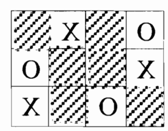
d3.


6d. Although there are four non death cells in board (d), there are only three possible pairs of moves for the third round. In each of these, only death cells remain after the third round. Beth will win in the fourth round.

e.


6e. In board (e) rounds three and four can be played in any order but the results are the same. Beth will win in the fifth round.

6f., 6g. Only death cells remain in boards (e) and (f). Therefore, Beth will win in round three on both of these boards.

h.


6h. Game board (h) at the end of round three is at the right. Again, all of the remaining cells are death cells so Beth will win in round four.

Therefore, Beth will always win if she uses this strategy!

7. As was shown in problem A2, case 2, there are nine ways for Ann to put two X's on the game board while playing smart:

X			X

a.

X			
			X

b.

X			
			X

c.

X			
		X	

d.

X			
	X		

e.

	X		
X			

f.

	X		
			X

g.

	X		
		X	

h.

X			X

i.

8. In board (a), Beth could put a O in cells 2, 5, 6, 9, or 10 to produce non-equivalent boards. [5]

In board (b), Beth could put a O in any of the open cells or an X in cell 10. [11]

In board (c), Beth could put a O in cells 2, 3, 4, 7, or 8. [5]

In board (d), Beth could put a O in any of the open cells. [10]

In board (e), Beth could put a O in any of the open cells or an X in cell 8. However, having an X's in cells 1, 8, and 10 has already been counted in board (b) [10]

In board (f), Beth could put a O in any of the open cells or an X in cells 8 or 11. [12]

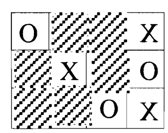
In board (g), Beth could put a O in any of the open cells or an X in cells 5, 9, or 11. However, having X's in cells 2, 8, and 5 or having X's in cells 2, 8, and 9 has already been counted in board (f) and having X's in cells 2, 8, and 11 was already counted in board (b). [10]

In board (h), Beth could put a O in cells 1, 5, 6, 9, 10 or an X in cell 5. However, having X's in cells 2, 11, and 5 has already been counted in board (f) [5]

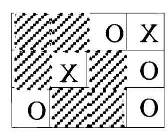
In board (i), Beth could put a O in cells 1, 2, or 6. or an X in cell 2. However, having an X's in cells 5, 8, and 2 has already been counted in board (f) [3]

The total number of nonequivalent game boards after the first round is **71**.

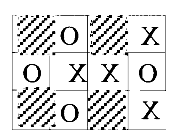
9a. As shown in board (d) above, Beth can only use O's in any of the ten open cells.



9b. On her second move, Beth must put an O in cell 8 and Ann can put a X in cell 12, leaving only death cells.

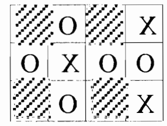


9c. On her second move, Beth must put an O in either cell 8 or 12. In either case, Ann responds by putting an O in the other one, leaving only death cells.



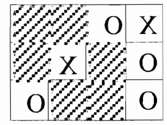
9d. On her second move, Beth must put an O in either cell 5 or cell 8. Whichever one she choses Ann will put an O in the other cell, leaving two non death cells, 7 and 12, which must be filled with X's. Whichever one Beth choses for her X, Ann will put an X in the other, leaving only death cells.

9e. Two cells, 8 and 10, are available. Beth must put an O in one of them and Ann counters with an O in the other. This frees up cell 12 in which Beth must put an X. This in turn frees up cell 2 in which Ann puts an O, leaving only death cells remaining open on the board.



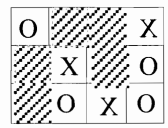
9f. Although there are five cells available, cells 1, 3, 5, 8, and 9, only cell 9 guarantees Ann a win!

Case 1 Show cell 9 guarantees a win. If Ann puts an O in cell 9, only cells 3 and 8 are available. Beth must put an O in one of them; Ann will put an O in the other, leaving only death cells.

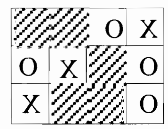


Case 2 Show the other four starts could force Ann to lose.

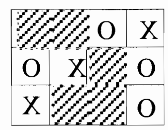
a) If Ann puts an O in cell 1, Beth could counter with an O in cell 8,which would force Ann to put an X in cell 11. Beth could counter with an O in cell 10, leaving only death cells for Ann.



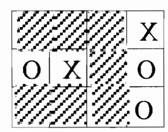
b) If Ann puts an O in cell 3, Beth counters with an X in cell 9, leaving only cells 5 and 8 available. Ann must put an O in one of them and Beth counters with an O in the other, leaving only death cells for Ann.



c) If Ann puts an O in cell 5, Beth counters with an O in cell 8, forcing Ann to put an O in cell 3. Beth counters with an X in cell 9, leaving only death cells for Ann.

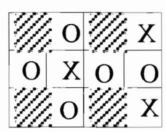


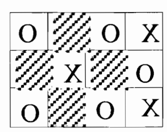
d) If Ann puts an O in cell 8, Beth counters with an O in cell 5, leaving only death cells for Ann



9g. Case 1 Ann places an X in cell 12.

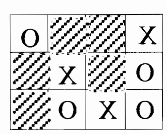
Beth must counter with an O in one of the eight remaining circular cells. Ann counters with an O in the "opposite" cell of the circle. Beth now must put an O in one of the two remaining safe cell; Ann counters with an O in the "opposite" safe cell, leaving only death cells. Ann wins.





Case 2 Ann place an O in cell 12.

Beth counters with an O in cell 1. Ann must play an X in cell 11 and Beth counters with an O in cell 10, leaving only death cells. Beth wins.



10. The easiest way to think of the game board on a torus is to imagine that the board repeats itself on each side and each corner. From the original board, if Ann plays an X in cell 1, Beth cannot play an X in cells 2, 3, 5, 9, 6, or 11. Looking at the expanded game board, cells 7, 12, and 1 form a downward diagonal of three cells and cells 8, 1 and 10 form an upward diagonal of three cells. Therefore, Beth cannot play an X in cells 7, 8, 10, or 12. Cell 4 is on the left of cell 1 and hence is not a smart play for an X. Therefore, Beth must play an O.

1	2	3	4	1	2	3	4	1	2	3	4
5	6	7	8	5	6	7	8	5	6	7	8
9	10	11	12	9	10	11	12	9	10	11	12
1	2	3	4	**1**	**2**	**3**	**4**	1	2	3	4
5	6	7	8	**5**	**6**	**7**	**8**	5	6	7	8
9	10	11	12	**9**	**10**	**11**	**12**	9	10	11	12
1	2	3	4	1	2	3	4	1	2	3	4
5	6	7	8	5	6	7	8	5	6	7	8
9	10	11	12	9	10	11	12	9	10	11	12

11. In problem D1 above, it was shown that every cell is in a row of three with every other cell. Prior to Ann's second turn, the board contains an X and an O. Since every cell is in a row of three with the cell containing the X and every cell is in a row of three with the cell containing the O, all remaining cells are death cells and Beth will win.

Three Addition Problems

The Definitions

Consider the following alphameric addition problem:

$$\begin{array}{r} A\,B\,C \\ +\underline{D\,E\,F} \\ G\,H\,I \end{array}$$

where $A\,B\,C$, $D\,E\,F$, and $G\,H\,I$ represent three three-digit numbers with $A\,B\,C + D\,E\,F = G\,H\,I$. In this problem $A, B, C, D, E, F, G, H,$ and I are distinct digits from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

$$\begin{array}{r} 2\,3\,5 \\ +\underline{7\,4\,6} \\ 9\,8\,1 \\ -\ i\ - \end{array}$$ is a solution to this problem.

Likewise, $\begin{array}{r} 2\,4\,5 \\ +\underline{7\,3\,6} \\ 9\,8\,1 \\ -\ ii\ - \end{array}$, $\begin{array}{r} 3\,5\,2 \\ +\underline{4\,6\,7} \\ 8\,1\,9 \\ -\ iii\ - \end{array}$, $\begin{array}{r} 2\,1\,5 \\ +\underline{4\,7\,8} \\ 6\,9\,3 \\ -\ iv\ - \end{array}$ are also solutions to this problem. Notice that in each of these solutions two of the addend digits total more than ten and so a <u>carry digit</u> must be added to the column on its left.

Although each of these three new solutions is <u>unique</u> from the original solution *i*, only the last one *iv* will be considered a solution <u>independent</u> of the original solution. In this problem, two unique solutions, $a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1, i_1$ and $a_2, b_2, c_2, d_2, e_2, f_2, g_2, h_2, i_2$, are considered <u>independent</u> if and only if the set $\{\{a_1, d_1\}, \{b_1, e_1\}, \{c_1, f_1\}\} \neq$ the set $\{\{a_2, d_2\}, \{b_2, e_2\}, \{c_2, f_2\}\}$ (Note:. The order elements are listed in a set is insignificant.) For example, in the solutions cited, solutions *i, ii,* and *iii* are not independent because the set $\{\{2, 7\}, \{3, 4\}, \{5, 6\}\}$ = the set $\{\{2, 7\}, \{4, 3\}, \{5, 6\}\}$ = the set $\{\{3, 4\}, \{5, 6\}, \{2, 7\}\}$, but solution *i* is independent of solution *iv* because the set $\{\{2, 7\}, \{3, 4\}, \{5, 6\}\} \neq$ the set $\{\{2, 4\}, \{1, 7\}, \{5, 8\}\}$.

So how many unique solutions are there to this problem? In this problem, $A + B + C + D + E + F + G + H + I = 45$ and by considering whether there are zero, one, or two carry digits, it can be shown that $G + H + I = 18$. (Can you verify this?) Therefore, $\{G, H, I\}$ must be one of the following: $\{1, 8, 9\}$, $\{2, 7, 9\}$, $\{3, 6, 9\}$, $\{3, 7, 8\}$, $\{4, 5, 9\}$, $\{4, 6, 8\}$, or $\{5, 6, 7\}$. (Can you verify this?)

Each of these produces one, two or three independent solutions, totaling 21 independent solutions. (This also could be verified by examining each of the above seven cases. For example, $\{1, 8, 9\}$ produces these three independent solutions: $234 + 657 = 891$, $235 + 746 = 981$, and $324 + 657 = 981$. Notice that in these solutions, $\{\{2,6\},\{3,5\},\{4,7\}\} \neq \{\{2,7\},\{3,4\},\{5,6\}\} \neq \{\{3,6\},\{2,5\},\{4,7\}\}$.)

Furthermore, each of these twenty-one independent solutions produces eight unique solutions since the order of each of the three pairs of addend digits can be permuted in $(2)^3 = 8$ ways (Compare solutions *i* and *ii* on page 1). In addition, in each of these eight unique solutions, the first column can always be interchanged with the last pair of columns (Compare solutions *i* and *iii* on page 1), producing another eight unique solutions.

Therefore, there is a total of $21(8)(2) = 336$ unique solutions to this problem!

Although alphameric problems usually have a solver trying to find a single unique solution, the alphamerics in this ARML Power Contest have many unique solutions where the solvers must determine a method to find the total number of unique solutions.

ARML Power Contest – November 2002 – Three Addition Problems

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The Problems

Since the order in which elements are listed in a set is irrelevant, for ease of correcting and checking independence, please always list the elements of a set in numerical order, smallest to largest.

1. Consider the following alphameric addition problem:

$$\begin{array}{r} ABC \\ +DEF \\ \hline GHIJ \end{array}$$

where ABC and DEF represent two three-digit numbers with $ABC + DEF = GHIJ$, a four-digit number and $A, B, C, D, E, F, G, H, I,$ and J are distinct digits from the set $\{0,1,2,3,4,5,6,7,8,9\}$. (Note: $A, D,$ or G cannot be zero.) In this problem, two unique solutions, $a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1, i_1, j_1$ and $a_2, b_2, c_2, d_2, e_2, f_2, g_2, h_2, i_2, j_2$, are considered independent if $\{\{a_1, d_1\}, \{b_1, e_1\}, \{c_1, f_1\}\} \neq$ $\{\{a_2, d_2\}, \{b_2, e_2\}, \{c_2, f_2\}\}$.

1a. Make two quick observations about the digits 0 and 1. Justify your conjectures.

1b. It is obvious that there is one carry digit from the hundreds column to the thousands column. Prove there can be either one carry digit or three carry digits, but not two carry digits, in this problem. Then show that the only possible sets equal to {G, H, I, J} are {0, 1, 2, 6}, {0, 1, 3, 5}, and {0, 1, 8, 9}.

1c. For each of the three sets above, find all independent solutions, i.e., find the possible sets for {{A, D}, {B, E}, {C, F}}.

1d. Determine the total number of unique solutions to this problem. Justify your answer.

2. Consider the following alphameric addition problem:

$$\begin{array}{r} A \\ BC \\ +DEF \\ \hline GHIJ \end{array}$$

where A is a one-digit number, BC a two-digit number, and DEF a three-digit number with $A + BC + DEF = GHIJ$, a four-digit number and $A, B, C, D, E, F, G, H, I,$ and J are distinct digits from the set $\{0,1,2,3,4,5,6,7,8,9\}$. (Note: $B, D,$ or G cannot be zero.) In this problem, two unique solutions, $a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1, i_1, j_1$ and $a_2, b_2, c_2, d_2, e_2, f_2, g_2, h_2, i_2, j_2$, are considered independent if $\{\{a_1, c_1, f_1\}, \{b_1, e_1\}, \{d_1\}\} \neq \{\{a_2, c_2, f_2\}, \{b_2, e_2\}, \{d_2\}\}$.

2a. Determine all the possible four-element sets that are equal to {G, H, I, J}. Show your work.

2b. For each of the sets from 2a above, determine all the independent solutions, i.e., find the sets which would equal {{A, C, F}, {B,E}}.

2c. Determine the number of unique solutions to this alphameric problem. Justify your answer.

3. Consider the following alphameric addition problem:

$$\begin{array}{r} A_{twelve} \\ B_{twelve} \\ C\,D_{twelve} \\ F\,G_{twelve} \\ +\,H\,I\,J_{twelve} \\ \hline K\,L\,M_{twelve} \end{array}$$

where A and B are one-digit base twelve numbers, $C\,D$ and $F\,G$ are two-digit base twelve numbers, and $H\,I\,J$ and $K\,L\,M$ are three-digit base twelve numbers with $A + B + C\,D + F\,G + H\,I\,J = K\,L\,M$ and $A, B, C, D, F, G, H, I, J, K, L,$ and M are distinct digits from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E\}$, where T represents the digit ten in the base twelve system and E represents the digit eleven. (Note: $C, F, H,$ or K cannot be zero.) In this problem two unique solutions, $a_1, b_1, c_1, d_1, f_1, g_1, h_1, i_1, j_1, k_1, l_1, m_1$ and $a_2, b_2, c_2, d_2, f_2, g_2, h_2, i_2, j_2, k_2, l_2, m_2$ are considered <u>independent</u> if $\{\{a_1, b_1, d_1, g_1, j_1\}, \{c_1, f_1, i_1\}, \{h_1\}\} \neq \{\{a_2, b_2, d_2, g_2, j_2\}, \{c_2, f_2, i_2\}, \{h_2\}\}$.

3a. It can easily be shown that the sum of the carry digits must be even in this problem. Using this fact, find the **twenty** possible sets equal to {K, L, M}. Show your work.

3b. Pick **one** of the {K, L, M} sets from 3a and list all the independent solutions for that set. Then determine the number of unique solutions for this set of independent solutions.

BONUS (Not part of this competition) Prove there are 108,960 solutions to this problem!

The Solutions

1a. G must be 1. It must be produced as a carry digit and A + B is at most 17, causing a carry digit of 1. C or F cannot be 0. If C = 0, then F = J and if F = 0, then C = J. A or D cannot be 0. If A = 0, then D must be 9 and H would have to be at least 2, implying that B + E > 30, an impossibility. There is a similar argument if D = 0.

1b. Case 1 Only 1 carry digit: $G = 0$, $C + F = J$, $B + E = I$, and $A + D = 10 + H$. Since the carry digit is produced in the hundreds column, H = 0. $A + B + C + D + E + F + G + I + J = 45$. Substituting produces: $10 + I + J + 0 + 1 + I + J = 45 \Rightarrow 2J + 2I + 11 = 45 \Rightarrow J + I = 17$. Therefore, $\{I, J\} = \{8, 9\}$ and $\{G, H, I, J\} = \{0, 1, 8, 9\}$

Case 2 Exactly two carry digits: Whether the second carry is produced in the units or the tens column makes no difference. $G = 1$, $C + F = 10 + J$, $B + E + 1 = I$, and $A + D = 10 + H$. Since

$A + B + C + D + E + F + G + I + J = 45 \Rightarrow 10 + H + I - 1 + 10 + J + 1 + H + I = 45 \Rightarrow$

$2H + 2I + 2J + 20 = 45 \Rightarrow H + I + J = 12.5$.....impossible!

Case 3 Three carry digits: $G = 1, C + F = 10 + J$, $B + E + 1 = 10 + I$, and $A + D + 1 = 10 + H$. Since $A + B + C + D + E + F + G + I + J = 45 \Rightarrow 9 + H + 9 + I + 10 + J + H + I = 45 \Rightarrow 2H + 2I + 2J + 29 = 45 \Rightarrow H + I + J = 8$. One of these three variables must be 0 because $2 + 3 + 4 > 8$. Therefore, $\{G, H, I, J\} = \{0, 1, 2, 6\}$ or $\{G, H, I, J\} = \{0, 1, 3, 5\}$.

1c. Case 1 $\{G, H, I, J\} = \{0, 1, 8, 9\}$

This case produces two possible values for the four-digit number $GHIJ$, either 1089 or 1098 with $\{A, B, C, D, E, F\} = \{2, 3, 4, 5, 6, 7\}$. If $GHIJ = 1089$, then $C + F = 9, B + E = 8$, and $A + D = 10$. Therefore, $\{C, F\} = \{2, 7\}, \{3, 6\}$, or $\{4, 5\}$.

Case 1a: $\{C, F\} = \{2, 7\}$, then $\{B, E\} = \{3, 5\}$ and $\{A, D\} = \{4, 6\}$, producing this solution:

$432 + 657 = 1089$ and $\{\{A, D\}, \{B, E\}, \{C, F\}\} = \{\{4, 6\}, \{3, 5\}, \{2, 7\}\}$.

Case 1b: $\{C, F\} = \{3, 6\}$, then none of the remaining numbers add up to 8. Therefore, $\{C, F\} \neq \{3, 6\}$.

Case 1c: $\{C, F\} = \{4, 5\}$, then $\{B, E\} = \{2, 6\}$ and $\{A, D\} = \{3, 7\}$, producing this solution:

$324 + 765 = 1089$ and $\{\{A, D\}, \{B, E\}, \{C, F\}\} = \{\{3, 7\}, \{2, 6\}, \{4, 5\}\}$.

All solutions for 1098 can be found by interchanging sets $\{C, F\}$ and $\{B, E\}$.

Case 2 $\{G, H, I, J\} = \{0, 1, 2, 6\}$

This case produces six possible values for the four-digit number $GHIJ$, either 1026, 1062, 1206, 1260, 1602 or 1620 with $\{A, B, C, D, E, F\} = \{3, 4, 5, 7, 8, 9\}$.

Case 2a: If $GHIJ$ = 1026, then $\{A, B, C, D, E, F\} = \{3, 4, 5, 7, 8, 9\}$. $C + F = 16, B + E = 15$, and $A + D = 9$, producing this solution: $437 + 589 = 1026$ and $\{\{A, D\}, \{B, E\}, \{C, F\}\} = \{\{4, 5\}, \{3, 8\}, \{7, 9\}\}$.

All solutions for 1206 can be found by interchanging sets $\{A, D\}$ and $\{B, E\}$.

Case 2b: If $GHIJ$ = 1062, then $C + F = 12, B + E = 15$, and $A + D = 9$. Therefore, $\{C, F\} = \{3, 9\}, \{4, 8\}$, or $\{5, 7\}$.

Case i: $\{C, F\} = \{3, 9\}$, then $\{B, E\} = \{7, 8\}$ and $\{A, D\} = \{4, 5\}$, producing this solution: $473 + 589 = 1062$ and $\{\{A, D\}, \{B, E\}, \{C, F\}\} = \{\{4, 5\}, \{7, 8\}, \{3, 9\}\}$.

Case ii: $\{C, F\} = \{4, 8\}$ or $\{5, 7\}$, then none of the remaining numbers add up to 15. Therefore, $\{C, F\} \neq \{4, 8\}$ or $\{5, 7\}$.

All solutions for 1602 can be found by interchanging sets $\{A, D\}$ and $\{B, E\}$.

Case 2c: If $GHIJ$ = 1260, then $C + F = 10, B + E = 15$, and $A + D = 11$. Therefore, $\{C, F\} = \{3, 7\}$ but none of the remaining pairs of digits add up to 11 or 15. Likewise, there are no solutions for 1620.

Case 3 $\{G, H, I, J\} = \{0, 1, 3, 5\}$

This case produces six possible values for the four-digit number $GHIJ$, either 1035, 1053, 1305, 1350, 1503, 1530 with $\{A, B, C, D, E, F\} = \{2, 4, 6, 7, 8, 9\}$.

Case 3a: If $GHIJ$ = 1035, then $C + F = 15, B + E = 12$, and $A + D = 9$. Therefore, $\{A, D\} = \{2, 7\}$ and so $\{B, E\} = \{4, 8\}$ and $\{C, F\} = \{6, 9\}$, producing this solution: $246 + 789 = 1035$ and $\{\{A, D\}, \{B, E\}, \{C, F\}\} = \{\{2, 7\}, \{4, 8\}, \{6, 9\}\}$. .

All solutions for 1305 can be found by interchanging sets $\{A, D\}$ and $\{B, E\}$.

Case 3b: If $GHIJ$ = 1053, then $C + F = 13, B + E = 14$, and $A + D = 9$. Therefore, $\{A, D\} = \{2, 7\}$ and so $\{B, E\} = \{6, 8\}$ and $\{C, F\} = \{4, 9\}$, producing this solution: $264 + 789 = 1053$ and $\{\{A, D\}, \{B, E\}, \{C, F\}\} = \{\{2, 7\}, \{6, 8\}, \{4, 9\}\}$.

All solutions for 1503 can be found by interchanging sets $\{A, D\}$ and $\{B, E\}$.

Case 3c: If $GHIJ$ = 1350, then $C + F = 10, B + E = 14$, and $A + D = 12$. Therefore, $\{A, D\} = \{4, 8\}$ but none of the remaining pairs of digits add up to 10 or 14. Likewise, there are no solutions for 1530.

1d. In 1c six independent solutions were found:

432	324	437	473	264	246
657	765	589	589	789	789
1089	1098	1206	1062	1053	1035

In each solution two columns can be interchanged to produce another set of solutions:

423	342	347	743	624	426
675	756	859	859	879	879
1098	1098	1206	1602	1503	1305

In each of these ten solutions the pairs of addend digits in any column can be permuted to produce 23 = 8 unique solutions. Therefore, there are $6 \times 2 \times 8 = 96$ unique solutions.

2a. It is easy to verify that the following must be true: $G = 1, H = 0$, and $D = 9$. As in problem 1, the carry digits must total to an odd number. $A + C + F \geq 2 + 3 + 4 = 9$, but $J \neq 9$. Therefore, $A + C + F \geq 10$ and $B + E + carry\ digit \geq 10$. So $A + C + F = J + 10$, $B + E + 1 = I + 10$, $G = 1$, $H = 0$, and $D = 9$. Since $A + B + C + D + E + F + G + I + J = 45 \Rightarrow J + 10 + I + 9 + 9 + 1 + 0 + I + J = 45 \Rightarrow 2J + 2I + 29 = 45$ and so $J + I = 8$.

Therefore, $\{G, H, I, J\} = \{0, 1, 2, 6\}$ *or* $\{0, 1, 3, 5\}$.

2b. <u>Case 1</u> $\{G, H, I, J\} = \{0, 1, 2, 6\}$

This case produces two possible values for the four-digit number $GHIJ$, either 1026 or 1062 with $\{A, B, C, E, F\} = \{3, 4, 5, 7, 8\}$ and $D = 9$.

<u>case 1a</u>: If $GHIJ$ = 1026, then $A + C + F = 16$ and $B + E = 11$. Therefore, $\{B, E\} = \{3, 8\}$ or $\{4, 7\}$.

If $\{B, E\} = \{3, 8\}$, then $\{A, C, F\} = \{4, 5, 7\}$, producing the solution, $4 + 35 + 987 = 1026$, with $\{\{A, C, F\}, \{B, E\}\} = \{\{4, 5, 7\}, \{3, 8\}\}$.

If $\{B, E\} = \{4, 7\}$, then $\{A, C, F\} = \{3, 5, 8\}$, producing the solution, $3 + 45 + 987 = 1026$, with $\{\{A, C, F\}, \{B, E\}\} = \{\{3, 5, 8\}, \{4, 7\}\}$.

case b: If $GHIJ$ = 1062, then $A + C + F = 12$ and $B + E = 15$. In this case $\{B, E\} = \{7, 8\}$ and $\{A, C, F\} = \{3, 4, 5\}$, producing the solution, $3 + 74 + 985 = 1062$, with $\{\{A, C, F\}, \{B, E\}\} = \{\{3, 4, 5\}, \{7, 8\}\}$.

<u>Case 2</u> $\{G, H, I, J\} = \{0, 1, 3, 5\}$

This case produces two possible values for the four-digit number $GHIJ$, either 1035 or 1053 with $\{A, B, C, E, F\} = \{2, 4, 6, 7, 8\}$ and $D = 9$.

<u>case 2a</u>: If $GHIJ$ = 1026, then $A + C + F = 15$ and $B + E = 12$. Therefore, $\{B, E\} = \{4, 8\}$ and $\{A, C, F\} = \{2, 6, 7\}$, producing the solution, $2 + 46 + 987 = 1035$, with $\{\{A, C, F\}, \{B, E\}\} = \{\{2, 6, 7\}, \{4, 8\}\}$.

case 2b: If $GHIJ = 1062$, then $A + C + F = 13$ and $B + E = 14$. Therefore, $\{B, E\} = \{6, 8\}$ and $\{A, C, F\} = \{2, 4, 7\}$, producing the solution, $2 + 64 + 987 = 1053$, with $\{\{A, C, F\}, \{B, E\}\} = \{\{2, 4, 7\}, \{6, 8\}\}$.

2c. In each of the five independent solutions found in 2b., the digits A, C, and F can be permuted in $3! = 6$ ways while the digits B and E can be permuted in $2! = 2$ ways with each permutation producing a unique solution. Therefore, there are $5 \times 6 \times 2 = 60$ unique solutions.

3a. Case 1: If there are two unit carry digits, then $A + B + C + D + F + G + I + J + K + L + M = 66$ with $A + B + D + G + J = 12 + M$, $C + F + I + 1 = L + 12$. and $H + 1 = K$. Substituting, produces: $12 + M + 11 + L + K - 1 + K + L + M = 66 \Rightarrow 2M + 2L + 2K + 22 = 66 \Rightarrow M + L + K = 22$. Therefore, $\{K, L, M\} = \{1, T, E\}$, $\{2, 9, E\}$, $\{3, 8, E\}$, $\{3, 9, T\}$, $\{4, 7, E\}$, $\{4, 8, T\}$, $\{5, 6, E\}$, $\{5, 7, T\}$, $\{5, 8, 9\}$, or $\{6, 7, 9\}$.

Case 2: If there is a carry digit of 1 and a carry digit of 3, then $A + B + C + D + F + G + I + J + K + L + M = 66$ with $A + B + D + G + J = 36 + M$, $C + F + I + 3 = L + 12$, and $H + 1 = K$. Substituting, produces: $36 + M + 9 + L + K - 1 + K + L + M = 66 \Rightarrow 2M + 2L + 2K + 44 = 66 \Rightarrow M + L + K = 11$. Therefore, $\{K, L, M\} = \{0, 1, T\}$, $\{0, 2, 9\}$, $\{0, 3, 8\}$, $\{0, 4, 7\}$, $\{0, 5, 6\}$, $\{1, 2, 8\}$, $\{1, 3, 7\}$, $\{1, 4, 6\}$, $\{2, 3, 6\}$, or $\{2, 4, 5\}$.

3b. (The solution to this problem is long because I have included all ten cases.)

{I, T, E}: five independent solutions:

7	8	8	8	8
6	6	6	7	7
85	54	35	25	36
43	73	24	43	42
<u>902</u>	<u>902</u>	<u>970</u>	<u>960</u>	<u>950</u>
T1E	T1E	T1E	T1E	T1E

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 2! ways in the first two solutions and 3! ways in the last three solutions. Therefore, there are $2 \times 2! \times 5! + 3 \times 3! \times 5! = 2640$ unique solutions.

{2, 9, E}: sixteen independent solutions:

8	7	7	T	6	7	8	7	T
5	6	6	5	4	6	6	6	6
74	84	T5	74	83	85	T4	T5	84
63	53	34	63	71	43	73	83	73
T08	T01	801	801	T50	T10	150	140	150
E29	E29	92E	92E	E92	E29	2E9	2E9	29E

6	T	8	8	T	8	8
4	6	6	7	7	6	7
T3	74	T5	T5	35	74	65
71	53	74	63	41	53	41
850	810	130	140	860	T10	T30
9E2	92E	29E	29E	92E	E29	E29

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 2! ways in the first four solutions and 3! ways in the last twelve solutions. Therefore, there are $4 \times 2! \times 5! + 12 \times 3! \times 5! = 9600$ unique solutions.

{3, 8, E}: twelve independent solutions:

7	7	9	T	7	7	9	9
6	6	6	6	5	6	5	6
94	T5	T5	94	92	95	74	74
52	94	42	52	61	42	62	51
T01	201	701	701	T40	T10	T10	T20
E38	38E	83E	83E	E83	E38	E38	E38

9	T	9	T
6	5	7	6
T4	94	T6	95
71	71	51	42
250	260	240	710
3E8	3E8	38E	83E

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 2! ways in the first four solutions and 3! ways in the last eight solutions. Therefore, there are $4 \times 2! \times 5! + 8 \times 3! \times 5! = 6720$ unique solutions.

{4, 7, E}: fourteen independent solutions:

8	9	6	8	8	9	9	T
5	6	5	5	6	5	6	6
93	T5	93	92	93	83	83	95
62	82	82	61	52	62	51	82
T01	301	T10	T30	T10	T10	T20	310
E47	47E	E74	E74	E47	E47	E47	47E

9	T	T	T	8	T
8	6	8	9	5	3
T5	92	93	83	T2	92
61	81	52	51	91	81
320	350	610	620	630	650
47E	4E7	74E	74E	7E4	7E4

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 2! ways in the first two solutions and 3! ways in the last twelve solutions. Therefore, there are $2 \times 2! \times 5! + 12 \times 3! \times 5! = 9120$ unique solutions.

{5, 6, E}: eleven independent solutions:

7	8	T	9	8	9	8	9
4	4	7	8	4	4	7	4
93	93	93	T3	93	83	92	83
82	72	82	72	72	71	41	72
T01	T01	401	401	T10	T20	T30	T10
E65	E56	56E	56E	E65	E65	E56	E56

T	T	8
8	9	7
93	83	T2
72	71	91
410	420	430
56E	56E	5E6

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 2! ways in the first four solutions and 3! ways in the last seven solutions. Therefore, there are $4 \times 2! \times 5! + 7 \times 3! \times 5! = 6000$ unique solutions.

{3, 9, T}: nine independent solutions:

7	7	E	E	8	E	7	8	E
5	5	5	6	7	5	6	7	6
E2	E4	74	74	E5	84	E5	E6	84
61	26	62	51	61	71	84	51	71
840	810	810	820	240	260	210	240	250
9T3	93T	93T	93T	3T9	3T9	39T	39T	39T

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 3! ways. Therefore, there are $9 \times 3! \times 5! = 6480$ unique solutions.

{4, 8, T}: thirteen independent solutions:

E	6	7	7	7	E	6	9
5	5	5	6	6	5	5	6
93	E3	E3	E2	E5	73	E3	E5
62	72	61	51	32	61	92	32
701	910	920	930	910	920	710	710
84T	T84	T84	T84	T48	T48	8T4	84T

9	9	E	7	E
6	7	6	6	6
15	25	13	15	52
72	61	52	92	71
3E0	3E0	790	3E0	390
48T	48T	84T	4T8	4T8

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 2! ways in the first solution and 3! ways in the last twelve solutions. Therefore, there are $1 \times 2! \times 5! + 12 \times 3! \times 5! = 8880$ unique solutions.

{5, 7, T}: seventeen independent solutions:

8	8	E	8	8	E	8	9
4	6	3	6	6	4	4	4
E3	E2	82	E3	E4	83	E3	E3
62	41	61	42	31	61	92	81
910	930	940	910	920	920	610	620
T75	T75	T75	T57	T57	T57	7T5	7T5

E	9	9	E	8	9	9	E	E
3	8	8	8	6	6	8	6	8
92	E3	E4	92	E3	E3	E3	93	92
81	42	31	41	92	81	62	82	61
640	610	620	630	410	420	410	410	430
7T5	75T	75T	75T	5T7	5T7	57T	57T	57T

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 3! ways. Therefore, there are $17 \times 3! \times 5! = 12240$ unique solutions.

{5, 8, 9}: twelve independent solutions:

E	T	T	T	E	T	T	E
4	6	6	4	3	6	7	6
T3	E3	E4	E2	T2	E3	E2	T2
62	42	31	61	61	71	61	71
701	710	720	730	740	420	430	430
859	859	859	895	895	598	598	598

T	T	E	E
6	7	6	7
E3	E3	T3	T2
72	61	71	61
410	420	420	430
589	589	589	589

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 2! ways in the first solution and 3! ways in the last eleven solutions. Therefore, there are $1 \times 2! \times 5! + 11 \times 3! \times 5! = 8160$ unique solutions.

{6, 7, 9}: eleven independent solutions:

E	T	T	E	T	T	E	E
4	4	5	4	4	5	4	5
T3	E3	E2	T2	E3	E3	T3	T2
82	51	41	51	52	41	51	41
501	820	830	830	810	820	820	830
679	976	976	976	967	967	967	967

T	E	T
4	4	8
E3	T3	E2
82	81	41
510	520	530
697	697	679

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 2! ways in the first solution and 3! ways in the last ten solutions. Therefore, there are $1 \times 2! \times 5! + 10 \times 3! \times 5! = 7440$ unique solutions.

{0, 1, T}: two independent solutions:

4	5
6	6
27	27
38	38
95E	94E
T10	T01

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 2! ways. There are $2 \times 3! \times 5! = 1440$ unique solutions.

{0, 2, 9}: seven independent solutions:

5	4	4	3	4	3	4
6	6	5	5	5	5	6
37	37	36	47	16	17	17
48	58	7T	6T	3T	4T	3T
1ET	1TE	18E	18E	87E	86E	85E
290	290	290	290	920	920	902

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 3! ways. There are $7 \times 3! \times 5! = 5040$ unique solutions.

{0, 3, 8}: ten independent solutions:

4	4	4	1	4	4	4	2	1	4
6	5	5	5	6	5	5	4	5	5
17	17	16	49	57	67	16	19	29	19
59	69	7T	6T	19	19	2T	5T	4T	2T
2ET	2TE	29E	27E	2ET	2TE	79E	76E	76E	76E
380	380	380	380	380	380	830	830	830	803

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 3! ways. There are $10 \times 3! \times 5! = 7200$ unique solutions.

{0, 4, 7}: nine independent solutions:

5	2	2	1	1	2	3	2
8	6	5	6	5	8	5	5
19	18	18	28	29	19	18	18
2T	59	6T	5T	6T	3T	29	3T
36E	3TE	39E	39E	38E	65E	6TE	69E
407	470	470	470	470	704	740	740

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 3! ways. There are $8 \times 3! \times 5! = 5760$ unique solutions.

{0, 5, 6}: two independent solutions:

2	1
7	7
18	28
39	39
<u>4ET</u>	<u>4TE</u>
560	560

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 3! ways. There are $2 \times 3! \times 5! = 1440$ unique solutions.

{0, 5, 6}: two independent solutions:

2	1
7	7
18	28
39	39
<u>4ET</u>	<u>4TE</u>
560	560

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 3! ways. Therefore, there are $2 \times 3! \times 5! = 1440$ unique solutions.

{1, 2, 8}: two independent solutions:

3	3
5	4
49	59
6T	6T
<u>70E</u>	<u>70E</u>
812	821

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted in only 2! ways (I must be 0). There are $2 \times 2! \times 5! = 480$ unique solutions.

{1, 3, 7}: six independent solutions:

5	5	4	4	4	4
8	6	6	5	5	5
49	48	58	69	29	29
6T	79	9T	8T	8T	8T
<u>20E</u>	<u>20E</u>	<u>20E</u>	<u>20E</u>	<u>60E</u>	<u>60E</u>
317	371	371	371	713	731

The digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 2! ways. There are $6 \times 2! \times 5! = 1440$ unique solutions.

{1, 4, 6}: nine independent solutions:

5	2	2	3	2	3	2	7	7
7	7	5	7	8	7	4	9	9
29	58	79	29	39	28	37	2T	2T
8T	T9	8T	8T	7T	E9	T8	5E	3E
<u>30E</u>	<u>30E</u>	<u>30E</u>	<u>50E</u>	<u>50E</u>	<u>50E</u>	<u>50E</u>	<u>380</u>	<u>580</u>
416	461	461	614	614	641	641	461	641

In the first seven solutions, the digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted only 2! ways. In the last two solutions, the digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 3! ways. There are $7 \times 2! \times 5! + 2 \times 3! \times 5! = 3120$ unique solutions.

{2, 3, 6}: ten independent solutions:

5	4	5	4	4	4	1	4	1	0
7	8	7	7	5	7	8	7	7	8
49	59	48	T8	79	18	49	E8	49	19
8T	7T	E9	59	8T	T9	7T	19	8T	4T
<u>10E</u>	<u>10E</u>	<u>10T</u>	<u>10E</u>	<u>10E</u>	<u>50E</u>	<u>50E</u>	<u>50T</u>	<u>50R</u>	<u>57E</u>
236	236	263	263	263	623	623	632	632	632

In the first nine solutions, the digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted only 2! ways. In the last solution, the digits in the units column can be permuted in 5! ways while the digits in the twelves column can be permuted 3! ways. There are $9 \times 2! \times 5! + 1 \times 3! \times 5! = 2880$ unique solutions.

{2, 4, 5}: six independent solutions:

6	3	6	3	6	1
7	8	7	7	7	7
38	69	38	69	18	69
T9	7T	E9	8T	T9	8T
<u>10E</u>	<u>10E</u>	<u>10T</u>	<u>10E</u>	<u>30E</u>	<u>30E</u>
245	245	254	254	425	452

There are $6 \times 2! \times 5! = 1440$ unique solutions. WHEW!!

Number Theoretic Functions

The Definitions, Symbolism, and Theorems

Number theory is a branch of mathematics that deals with the properties of positive integers. This problem will deal with six functions that are used to state some of these properties. They are defined as follows:

$\tau(n)$ = the number of positive integral divisors of n. For example, $\tau(12) = 6$ because 12 has six divisors, namely 1, 2, 3, 4, 6, and 12.

$\sigma(n)$ = the sum of the positive integral divisors of n. For example, $\sigma(12) = 28$ because $1 + 2 + 3 + 4 + 6 + 12 = 28$.

$\varphi(n)$ = the number of positive integers less than n and relatively prime to n. $\varphi(12) = 4$ because 1, 5, 7, 11 are relatively prime to 12. Two integers, a and b, are relatively prime if and only if their greatest common factor is 1 (or symbolically gcf$(a, b) = 1$ or simply $(a, b) = 1$).

The fundamental theorem of arithmetic states that any positive integer, n, can be written as a product of primes in only one way, i.e., $n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdot \ldots \cdot p_k^{a_k}$, where p_i's are primes and a_i's are positive integers. For example, $12 = 2^2 \cdot 3^1$.

$\lambda(n)$ = $(-1)^{a_1 + a_2 + \ldots + a_k}$ and $\lambda(1) = 1$. Therefore, $\lambda(12) = (-1)^{(2+1)} = -1$.

$\mu(n)$ = 0 if any a_k is greater than 1, otherwise $\mu(n) = (-1)^k$ and $\mu(1) = 1$. Therefore, $\mu(12) = 0$.

$\theta(n)$ = number of ordered pairs (a,b) such that $(a,b) = 1$ and $ab = n$. $\theta(12) = 4$ for the pairs (1, 12), (3, 4), (4, 3), and (12, 1) have no common factors. The pairs (2, 6) and (6, 2) are not relatively prime.

It is amazing that such functions with such distinctive properties can be related in interesting ways but later in this problem it will be shown that for any prime number, n,, $\sigma(n) + \varphi(n) + \lambda(n) + \mu(n) + \theta(n) = n \cdot \tau(n)$. But first, here are some symbols and fundamental theorems regarding these functions.

Symbols:

$d \mid n$ means d is a divisor of n.

ARML Power Contest – February 2003 – Number Theoretic Functions

$\sum_{d|n} f(d)$ means sum up all the values of $f(d)$ where $d \mid n$. For example, $\tau(n) = \sum_{d|n} 1$ and $\sigma(n) = \sum_{d|n} d$.

$\sum_{(d,n)=1} f(d)$ means sum up all the values of $f(d)$ where d is relatively prime to n. For example, $\varphi(n) = \sum_{(d,n)=1} 1$.

$\prod_{d|n} f(d)$ means multiply together all the values of $f(d)$ where $d \mid n$.

Useful Theorems:

1. A function, f, is called multiplicative if for any positive integer, n, where $n = a \cdot b$ and $(a,b) = 1$, $f(n) = f(a \cdot b) = f(a) \cdot f(b)$. All six functions are multiplicative. For example, $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$.

2. If a is a non-negative integer and p is a prime number, then ...
   a) $\tau(p^a) = a + 1$.
   b) $\sigma(p^a) = \dfrac{p^{a+1} - 1}{p - 1}$.
   c) $\varphi(p^a) = p^a\left(1 - \frac{1}{p}\right)$.
   d) $\lambda(p^a) = (-1)^a$.
   e) $\mu(p^a) = \begin{cases} -1, \text{if } a = 1 \\ 0, \text{otherwise} \end{cases}$.
   f) $\theta(p^a) = 2$.

3. If $n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdot \ldots \cdot p_k^{a_k}$, then ...
   a) $t(n) = (a_1 + 1)(a_2 + 1)\ldots(a_k + 1)$.
   b) $\sigma(n) = \prod_{p|n} \dfrac{p^{a+1} - 1}{p - 1}$.
   c) $\varphi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right)$.

In math class you probably learned about the ***composition*** of two functions, $f(g(x))$, sometimes denoted as $f \circ g(x)$. In number theory two functions can be combined by a method called the ***convolution*** of two functions, often denoted as $f * g(n)$. It is defined as $f * g(n) = \sum_{d|n} f(d) \cdot g\left(\frac{n}{d}\right)$.

The following chart may be useful in solving the problems of the contest.

Number Theoretic Function Table

n	*prime factorization*	$\tau(n)$	$\sigma(n)$	$\varphi(n)$	$\lambda(n)$	$\mu(n)$	$\theta(n)$
1		1	1	1	1	1	1
2	2	2	3	1	-1	-1	2
3	3	2	4	2	-1	-1	2
4	2^2	3	7	2	1	0	2
5	5	2	6	4	-1	-1	2
6	(2)(3)	4	12	2	1	1	4
7	7	2	8	6	-1	-1	2
8	2^3	4	15	4	-1	0	2
9	3^2	3	13	6	1	0	2
10	(2)(5)	4	18	4	1	1	4
11	11	2	12	10	-1	-1	2
12	$(2^2)(3)$	6	28	4	-1	0	4
13	13	2	14	12	-1	-1	2
14	(2)(7)	4	24	6	1	1	4
15	(3)(5)	4	24	8	1	1	4
16	2^4	5	31	8	1	0	2
17	17	2	18	16	-1	-1	2
18	$(2)(3^2)$	6	39	6	-1	0	4
19	19	2	20	18	-1	-1	2
20	$(2^2)(5)$	6	42	8	-1	0	4
21	(3)(7)	4	32	12	1	1	4
22	(2)(11)	4	36	10	1	1	4
23	23	2	24	22	-1	-1	2
24	$(2^3)(3)$	8	60	8	1	0	4

ARML Power Contest – February 2003 – Number Theoretic Functions

The Problems

Part 1

1. Find the value for each of the six functions when $n = 72$.

2. Prove, if a is a non-negative integer and p is a prime number , then $\sigma(p^a) = \dfrac{p^{a+1} - 1}{p-1}$.

3. For what values of n is $\varphi(n) = \frac{n}{2}$? Prove your conjecture.

4. For what values of n does $\varphi(n) = \varphi(2n)$? Prove your result.

Part 2

5. For each of the following expressions:
   i) Find its value when $n = 24$. Show your work.

   ii) Experiment with enough other values of n (No need to show your work here.) which will allow you to make a conjecture about the expression that is true for all n. (All conjectures will involve a single number theoretic function or a constant.)

      a. $\varphi * \tau(n)$.
      b. $\lambda * \theta(n)$.
      c. $\sum_{d \mid n} \tau(d^2)\mu\left(\frac{n}{d}\right)$.
      d. $\varphi * \sigma(n)$.
      e. $\sum_{d \mid n} (\tau(d))^3$.

6. For what values of n does $\varphi(n) + \sigma(n) = 2n$?

7. For what values of n does $\sum_{d \mid n} \lambda(d) = 1$?

Part 3

8. Find all values of n for which $\varphi(n) = 24$. Show your work.

9. If $\sigma(n) = 2n$, then n is called a **perfect number**.
   a. Show 6, 28, 496, 8128 are perfect.
   b. Show if $2^n - 1$ and n are prime, then $2^{n-1}(2^n - 1)$ is perfect.
   c. Show $\sum_{d \mid n} \frac{1}{d} = 2$ if and only if n is perfect.

10. Prove, if n is prime, then $\sigma(n) + \varphi(n) + \lambda(n) + \mu(n) + \theta(n) = n \cdot \tau(n)$.

The Solutions

Part 1

1. $72 = 2^3 \cdot 3^2$ and the factors of 72 are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.

$\tau(72) = (3+1)(2+1) = \mathbf{12}$, the number of factors of 72.

$\sigma(72) = \left(\frac{2^4-1}{2-1}\right)\left(\frac{3^3-1}{3-1}\right) = 15 \cdot 13 = \mathbf{195}$ or

$\sigma(72) = 1+2+3+4+6+8+9+12+18+24+36+72 = 195$.

$\varphi(72) = 2^3\left(1-\frac{1}{2}\right)3^2\left(1-\frac{1}{3}\right) = 8\left(\frac{1}{2}\right)9\left(\frac{2}{3}\right) = 4 \cdot 6 = \mathbf{24}$, the number of elements in the set

$\{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49, 53, 55, 59, 61, 65, 67, 71\}$.

$\lambda(72) = (-1)^{(3+2)} = (-1)^5 = \mathbf{-1}$.

$\mu(72) = \mathbf{0}$.

$\theta(72) = \mathbf{4}$, the number of elements in the set $\{(1, 72), (8, 9), (9, 8), (72, 1)\}$.

2. The only factors of p^a are $1, p, p^2, p^3, \ldots, p^a$.

Therefore, $\sigma(p^a) = 1 + p + p^2 + p^3 + \ldots + p^a$, an infinite geometric sequence, whose sum is $\frac{a_0(1-r^n)}{1-r}$.

Therefore, $\sigma(p^a) = \frac{1(1-p^{a+1})}{1-p} = \frac{p^{a+1}-1}{p-1}$.

3. $\varphi(n) = \frac{n}{2}$ iff n is a power of 2.

Proof: <== If n is a power of 2, then $n = 2^a$. $\varphi(2^a) = 2^a\left(1-\frac{1}{2}\right) = 2^a\left(\frac{1}{2}\right) = \frac{n}{2}$.

==> $2\varphi(n) = n$. Therefore, n has a factor of 2. Assume there are other prime factors, i.e.,

$2\varphi(n) = 2^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdot \ldots \cdot p_k^{a_k}$. Then

$2\left(2^{a_1}\left(\frac{1}{2}\right) \cdot p_2^{a_2}\left(1-\frac{1}{p_2}\right) \cdot p_3^{a_3}\left(1-\frac{1}{p_3}\right) \cdot \ldots \cdot p_k^{a_k}\left(1-\frac{1}{p_k}\right)\right) = 2^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdot \ldots \cdot p_k^{a_k}$.

$2^{a_1} \cdot \left(1-\frac{1}{p_2}\right) \cdot \left(1-\frac{1}{p_3}\right) \cdot \ldots \cdot \left(1-\frac{1}{p_k}\right) = 2^{a_1}$. If $p_2, p_3, \ldots, p_k$ exist then the left side is less than the right side...a contradiction. Therefore, n must have no prime factors other then 2.

4. $\varphi(n) = \varphi(2n)$ iff n is odd.

Proof: <== If n is odd then $(2, n) = 1$ and so $\varphi(2n) = \varphi(2) \cdot \varphi(n) = 1 \cdot \varphi(n) = \varphi(n)$.

==> Suppose $\varphi(n) = \varphi(2n)$ and n is even, i.e., $n = 2^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdot \ldots \cdot p_k^{a_k}$ and

$\varphi(n) = 2^{a_1}\left(\frac{1}{2}\right) \cdot p_2^{a_2}\left(1-\frac{1}{p_2}\right) \cdot p_3^{a_3}\left(1-\frac{1}{p_3}\right) \cdot \ldots \cdot p_k^{a_k}\left(1-\frac{1}{p_k}\right)$. Then $2n = 2^{a+1_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdot \ldots \cdot p_k^{a_k}$ and

$\varphi(2n) = 2^{a+1_1}\left(\frac{1}{2}\right) \cdot p_2^{a_2}\left(1-\frac{1}{p_2}\right) \cdot p_3^{a_3}\left(1-\frac{1}{p_3}\right) \cdot \ldots \cdot p_k^{a_k}\left(1-\frac{1}{p_k}\right)$, which equals

$\varphi(2n) = 2^{a+1}\left(\frac{1}{2}\right)\cdot p_2^{a_2}\left(1-\frac{1}{p_2}\right)\cdot p_3^{a_3}\left(1-\frac{1}{p_3}\right)\cdot \ldots \cdot p_k^{a_k}\left(1-\frac{1}{p_k}\right)$, which equals

$\varphi(2n) = 2\left(2^{a}\left(\frac{1}{2}\right)\cdot p_2^{a_2}\left(1-\frac{1}{p_2}\right)\cdot p_3^{a_3}\left(1-\frac{1}{p_3}\right)\cdot \ldots \cdot p_k^{a_k}\left(1-\frac{1}{p_k}\right)\right) = 2\varphi(n)$. But then $\varphi(2n) = 2\varphi(n)$, a contradiction.

Part 2

5a. $\varphi * \tau(24) = \sum_{d|24} \varphi(d)\cdot\tau\left(\frac{24}{d}\right)$. The divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.

So $\varphi * \tau(24) = \varphi(1)\cdot\tau(24) + \varphi(2)\cdot\tau(12) + \varphi(3)\cdot\tau(8) + \varphi(4)\cdot\tau(6) + \varphi(6)\cdot\tau(4) + \varphi(8)\cdot\tau(3) + \varphi(12)\cdot\tau(2) + \varphi(24)\cdot\tau(1)$.

$= 1(8) + 1(6) + 2(4) + 2(4) + 2(3) + 4(2) + 4(2) + 8(1)$.

$= 8 + 6 + 8 + 8 + 6 + 8 + 8 + 8 = \mathbf{60}$.

Therefore, $\boldsymbol{\varphi * \tau(n) = \sigma(n)}$. Other examples should verify this.

5b. $\lambda * \theta(24) = \lambda(1)\cdot\theta(24) + \lambda(2)\cdot\theta(12) + \lambda(3)\cdot\theta(8) + \lambda(4)\cdot\theta(6) + \lambda(6)\cdot\theta(4) + \lambda(8)\cdot\theta(3) + \lambda(12)\cdot\theta(2) + \lambda(24)\cdot\theta(1)$.

$= 1(4) + (-1)(4) + (-1)(2) + 1(4) + (-1)(2) + (-1)(2) + (-1)(2) + 1(1)$.

$= 4 - 4 - 2 + 4 - 2 + 2 - 2 + 1 = \mathbf{1}$.

Therefore, $\boldsymbol{\lambda * \theta(n) = 1}$. Other examples should verify this.

5c. $\sum_{d|24} \tau(d^2)\cdot\mu\left(\frac{24}{d}\right) = \tau(1)\cdot\mu(24) + \tau(4)\cdot\mu(12) + \tau(9)\cdot\mu(8) + \tau(16)\cdot\mu(6) + \tau(36)\cdot\mu(4) + \tau(64)\cdot\mu(3) + \tau(144)\cdot\mu(2) + \tau(576)\cdot\mu(1)$.

$= 1(0) + 3(0) + 3(0) + 5(1) + 9(0) + 7(-1) + 15(-1) + 21(1)$.

$= 0 + 0 + 0 + 5 + 0 - 7 - 15 + 21 = \mathbf{4}$.

Therefore, $\boldsymbol{\sum_{d|24} \tau(d^2)\cdot\mu\left(\frac{24}{d}\right) = \theta(n)}$. Other examples should verify this.

5d. $\varphi * \sigma(24) = \varphi(1)\cdot\sigma(24) + \varphi(2)\cdot\sigma(12) + \varphi(3)\cdot\sigma(8) + \varphi(4)\cdot\sigma(6) + \varphi(6)\cdot\sigma(4) + \varphi(8)\cdot\sigma(3) + \varphi(12)\cdot\theta(2) + \varphi(24)\cdot\sigma(1)$

$= 1(60) + 1(28) + 2(15) + 2(12) + 2(7) + 4(4) + 4(3) + 8(1)$

$= 60 + 28 + 30 + 24 + 14 + 16 + 12 + 8 = \mathbf{192}$.

Probably not enough to make a conjecture yet and so more examples are needed:

$\varphi * \sigma(1) = 1$, $\varphi * \sigma(2) = 4$, $\varphi * \sigma(3) = 6$, $\varphi * \sigma(4) = 12$, $\varphi * \sigma(5) = 10$, and

Therefore, $\boldsymbol{\varphi * \sigma(n) = n\cdot\tau(n)}$.

5e. $\sum_{d|24} [\tau(d)]^3 = 1^3 + 2^3 + 2^3 + 3^3 + 4^3 + 4^3 + 6^3 + 8^3 = \mathbf{900}$. Again probably not enough to make a conjecture and so more examples are needed: $\sum_{d|2} [\tau(d)]^3 = 1^3 + 2^3 = 9$, $\sum_{d|3} [\tau(d)]^3 = 1^3 + 2^3 = 9$, All the answers are squares...but squares of what? Observe that $\sum_{d|n}(\tau(d))^3 = \left(\sum_{d|n}\tau(d)\right)^2$.

6. Examples: $\varphi(2)+\sigma(2)=4$, $\varphi(3)+\sigma(3)=6$, $\varphi(4)+\sigma(4)=9\neq 8$, $\varphi(5)+\sigma(5)=10$, $\varphi(6)+\sigma(6)=14\neq 12$ ...If n is prime then $\varphi(n)+\sigma(n)=2n$.

7. Examples: $\sum_{d|1}\lambda(d)=1$, $\sum_{d|2}\lambda(d)=0$, $\sum_{d|3}\lambda(d)=0$, $\sum_{d|4}\lambda(d)=1$ ... If n is a square then $\sum_{d|n}\lambda(d)=1$.

Part 3

8. $\varphi(n)=24=n\prod_{p|n}\left(1-\frac{1}{p}\right)$

$$= p_1^{a_1}\cdot p_2^{a_2}\cdot p_3^{a_3}\cdot\ldots\cdot p_k^{a_k}\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\left(1-\frac{1}{p_3}\right)\cdot\ldots\cdot\left(1-\frac{1}{p_k}\right)$$

$$= p_1^{a_1-1}\cdot p_2^{a_2-1}\cdot p_3^{a_3-1}\cdot\ldots\cdot p_k^{a_k-1}(p_1-1)(p_2-1)(p_3-1)\cdot\quad\cdot(p_k-1)$$

Therefore, if p is a prime factor of n, then $(p-1)$ must be a divisor of 24 and so $(p-1)$ could be 1, 2, 3, 4, 6, 8, 12, or 24. Therefore, the only possible prime factors on could be 2, 3, 5, 7, or 13.

$\varphi(13)=12, \varphi(7)=6, \varphi(5)=4, \varphi(3)=2, \varphi(3^2)=6, \varphi(2)=1, \varphi(2^2)=2, \varphi(2^3)=4$, and $\varphi(2^4)=8$.

All other powers of these primes produce φ*–values* which are not divisors of 24. Since $\varphi(n)$ is multiplicative and equal to 24, the following are the only possible values for n :

$24=(12)(2)$:	$\varphi(13)\cdot\varphi(3)$	$\Rightarrow n=\mathbf{39}$
	$\varphi(13)\cdot\varphi(2^2)$	$\Rightarrow n=\mathbf{52}$
$24=(12)(2)(1)$:	$\varphi(13)\cdot\varphi(3)\cdot\varphi(2)$	$\Rightarrow n=\mathbf{78}$
$24=(8)(3)$:	not possible...all φ*–values* are even except $\varphi(1)$ and $\varphi(2)$.	
$24=(6)(4)$:	$\varphi(7)\cdot\varphi(5)$	$\Rightarrow n=\mathbf{35}$
	$\varphi(7)\cdot\varphi(2^3)$	$\Rightarrow n=\mathbf{56}$
	$\varphi(3^2)\cdot\varphi(5)$	$\Rightarrow n=\mathbf{45}$
	$\varphi(3^2)\cdot\varphi(2^2)$	$\Rightarrow n=\mathbf{72}$
$24=(6)(4)(1)$:	$\varphi(7)\cdot\varphi(5)\cdot\varphi(2)$	$\Rightarrow n=\mathbf{70}$
	$\varphi(3^2)\cdot\varphi(5))\cdot\varphi(2)$	$\Rightarrow n=\mathbf{90}$
$24=(6)(2)(2)$:	$\varphi(7)\cdot\varphi(3)\cdot\varphi(2^2)$	$\Rightarrow n=\mathbf{84}$

9a. $6=2\cdot 3$ and $(2,3)=1$, *so* $\sigma(6)=\sigma(2)\sigma(3)=3\cdot 4=12=2(6)$.

$28=4\cdot 7$ and $(4,7)=1$, so $\sigma(28)=\sigma(4)\sigma(7)=7\cdot 8=56=2(28)$.

$496 = 16 \cdot 31$ and $(16,31) = 1$, so $\sigma(496) = \sigma(16)\sigma(31) = 31 \cdot 32 = 992 = 2(496)$.

$8128 = 64 \cdot 127$ and $(64,127) = 1$, so $\sigma(8128) = \sigma(64)\sigma(127) = 127 \cdot 128 = 2(8128)$.

9b. $2^n - 1$ is odd and 2^{n-1} is even. Therefore, $(2^n - 1, 2^{n-1}) = 1$ and so $\sigma(2^{n-1}(2^n - 1)) = \sigma(2^{n-1}) \cdot \sigma(2^n - 1)$.

By Theorem 2b, $\sigma(2^{n-1}) = \frac{2^n - 1}{2 - 1}$. Since the only factors of a prime, p, are 1 and p, $\sigma(p) = p + 1$ and so $\sigma(2^n - 1) = 2^n$. Therefore, $\sigma(2^{n-1}(2^n - 1)) = (2^n - 1)(2^n) = 2(2^{n-1})(2^n - 1) = 2n$.

9c. <== If n is perfect, $\sigma(n) = 2n$. But $\sigma(n) = \sum_{d|n} d = \sum_{d|n} \frac{n}{d} = n\sum_{d|n} \frac{1}{d}$. Therefore, $2n = n\sum_{d|n} \frac{1}{d}$ and $2 = \sum_{d|n} \frac{1}{d}$.

==> $\sum_{d|n} \frac{1}{d} = 2$ and multiplying both sides by n produces: $n\sum_{d|n} \frac{1}{d} = 2n$, which equals $\sum_{d|n} \frac{n}{d} = 2n$ and so $\sigma(n) = 2n$.

10. If n is prime, it has only two divisors, 1 and n, so $\tau(n) = 2$ and $\sigma(n) = n + 1$. All the numbers less than n are relatively prime to n, and so $\varphi(n) = n - 1$ and $\lambda(n) = -1$, $\mu(n) = -1$, and $\theta(n) = 2$. Therefore, $\sigma(n) + \varphi(n) + \lambda(n) + \mu(n) + \theta(n) = n \cdot \tau(n)$ is equivalent to $(n + 1) + (n - 1) + (-1) + (-1) + 2 = n \cdot 2$ which is true. (Note: The converse is also true but more difficult to prove.)

Further Notes

- Euler went on to prove that all perfect numbers are of the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is prime. Primes of the form $2^n - 1$ are known as Mersenne primes. If $2^n - 1$ is prime then n must be prime but the converse is not true for $2^{67} - 1$ is composite. As of this writing there are only 39 known perfect numbers. In November 2001 it was proven that $2^{13466917} - 1$ is prime. Its corresponding perfect number has 8,107,892 digits! No one has yet found an odd perfect number but much is known about them! Euler proved that any prime factor of an odd perfect number must be of the form 4n +1 and more recently it has been shown that an odd perfect number must have at least eight prime factors, one being greater than 10^{18}.
- An $n - gon$ can be drawn using a compass and straightedge if and only if $\varphi(n)$ is a power of two.
- In Lure of the Integers, Joe Roberts states that $\sigma(n) < \frac{6\sqrt{n^3}}{\pi^2}$ for all values of n, except 2, 3, 4, 6, 8, and 12; $\tau(n) < \sqrt[3]{n^2}$ except when n is 2, 4, 6, 12; and $\varphi(n) > \sqrt{n}$ except for 2 and 6.
- More investigation and discussions about number theoretic functions can be found in An Introduction to Arithmetical Functions, by P. J. McCarthy.

Errors in Mathematical Reasoning

In this contest problem you will be exploring errors in mathematical reasoning that ultimately lead to correct solutions. Do the errant methods work some of the time, all of the time, or just in one isolated case?

The Problems

1. Sometimes fractions can be reduced by canceling out digits. For example, $\frac{16}{64} = \frac{1\not{6}}{\not{6}4} = \frac{1}{4}$.

   Find the other two fractions that can be reduced in a similar manner, i.e. $\frac{ab}{bc} = \frac{a\not{b}}{\not{b}c} = \frac{a}{c}$, where $\frac{a}{c}$ is a reduced fraction with $a < c$.

2. While the distributive property of multiplication over addition, $a \cdot (b + c) = a \cdot b + a \cdot c$, always works with any set of real numbers, the "other" distributive property of addition over multiplication, $a + (b \cdot c) = (a + b) \cdot (a + c)$, is less successful. The "other" distributive is valid for the following example: $.5 + (.2 \bullet .3) = (.5 + .2) \bullet (.5 + .3) = (.7) \bullet (.8) = .56$ and $.5 + .2 \bullet .3 = .5 + .06 = .56$. Under what conditions for $a, b,$ and c, does $a + (b \cdot c) = (a + b) \cdot (a + c)$?

3. Anna incorrectly solved the following inequality but came up with the correct answer:

$$|x| + |x - 1| < 2$$
$$|2x - 1| < 2 \quad (\text{ In general, } |a + b| \neq |a| + |b|)$$
$$-2 < 2x - 1 < 2$$
$$-1 < 2x < 3$$
$$\frac{-1}{2} < x < \frac{3}{2}$$

   Under what conditions for $a, b,$ and c with $a < b$ and $c > 0$, is $|x - a| + |x - b| < c$ the same as $|2x - (a + b)| < c$?

4. Ning was asked to solve the following problem:

   Solve for x: $\sqrt{x} + \sqrt{x - a} = 4$, where $a \geq 0$.

   Remembering the difference of squares identity, he proceeded as follows:

$$\left(\sqrt{x} + \sqrt{x - a}\right)\left(\sqrt{x} - \sqrt{x - a}\right) = x - (x - a) = a.$$

   Therefore, $\left(\sqrt{x} - \sqrt{x - a}\right) = \frac{a}{\left(\sqrt{x} + \sqrt{x - a}\right)}$.

   And so, $\left(\sqrt{x} - \sqrt{x - a}\right) = \frac{a}{4}$.

Adding $\left(\sqrt{x}+\sqrt{x-a}\right)=4$ and $\left(\sqrt{x}-\sqrt{x-a}\right)=\frac{a}{4}$, he got $2\sqrt{x}=4+\frac{a}{4}$

$$2\sqrt{x}=\frac{16+a}{4}$$

$$\sqrt{x}=\frac{16+a}{8}=2+\frac{a}{8}$$

$$x=\left(2+\frac{a}{8}\right)^2$$

When $a=4$, $x=\frac{25}{4}$, a correct solution. When $a=8, x=9$, and when $a=16, x=16$. Both of these are also correct solutions to this problem. However, when $a=32, x=36$ and this is not a correct solution to the problem! Where, if any place, is the flaw in Ning's solution?

5. Oftentimes, due to carelessness, $\log\left(b^n\right)$ and $\left(\log b\right)^n$ get interchanged in a problem. For what positive values of b and positive integer values of n will this interchange not create a wrong answer to the problem, i.e., under these conditions, when does $\log\left(b^n\right)=\left(\log b\right)^n$?

6. Simplify: $\dfrac{3\frac{2}{5}}{5\frac{2}{3}}$

Method 1: Change to improper fractions and divide:

$$\frac{3\frac{2}{5}}{5\frac{2}{3}}=\frac{\frac{17}{5}}{\frac{17}{3}}=\frac{17}{5}\div\frac{17}{3}=\frac{17}{5}\cdot\frac{3}{17}=\frac{3}{5}$$

Method 2: Eliminate fractions by errantly multiplying by 1 (in this case $\frac{15}{15}$):

$$\frac{3\frac{2}{5}}{5\frac{2}{3}}=\frac{3+\frac{2}{5}}{5+\frac{2}{3}}=\frac{3+\frac{2}{5}\cdot 15}{5+\frac{2}{3}\cdot 15}=\frac{3+6}{5+10}=\frac{9}{15}=\frac{3}{5}$$

Under what conditions does $\dfrac{a\frac{b}{c}}{d\frac{e}{f}}$ reduce to the same fraction using both methods?

7. To add fractions, sometimes Siu Hin errs and adds the numerators together and then adds the denominators together, which generally produces a wrong answer. However, he finds $\frac{6}{\sqrt{18}}+\frac{-5}{\sqrt{15}}=\frac{6+(-5)}{\sqrt{18}+\sqrt{15}}=\frac{1}{\sqrt{18}+\sqrt{15}}$ produces a correct solution! Under what conditions for $a, b, c,$ and d, does $\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$?

8. Paul used the following algorithm to factor a quadratic: ax^2+bx+c.

Remove the leading coefficient and multiply the constant term by it .

$$x^2 + bx + a\cdot c$$

Factor the resulting quadratic into linear factors.

$$(x - r)(x - s)$$

Divide both r and s by a.

$$\left(x - \frac{r}{a}\right)\left(x - \frac{s}{a}\right)$$

Reduce the fractions, if possible.

$$\left(x - \frac{r'}{a'}\right)\left(x - \frac{s''}{a''}\right)$$

Remove the fractions by multiplying the factors by a' and a'', respectively.

$$(a'x - r')(a''x - s'')$$

For example.

$$6x^2 - 7x - 3 \longrightarrow x^2 - 7x - 18 \longrightarrow (x - 9)(x + 2) \longrightarrow (x - \tfrac{9}{6})(x + \tfrac{2}{6}) \longrightarrow (x - \tfrac{3}{2})(x + \tfrac{1}{3}) \longrightarrow (2x - 3)(3x + 1)$$

Although this method is seemingly full of algebraic errors, it worked in this example! As a matter of fact, any factorable quadratic can be factored using this method!! Prove this is true.

9. Mr. Kilkelly wrote on the board a quadratic equation in the form $x^2 - Ax + B = 0$ and asked the students to quickly solve it. In copying the problem, Sindhuja erroneously transposed the two digits of B as well as the plus and minus signs. However, she still got one of the correct roots. If A and B are integers, what was this root? What were the two possible equations Mr. Kilkelly wrote on the board?
10. Using the formula for the sum of an infinite geometric series, Hwa-Sheng produced:

$x + x^2 + x^3 + x^4 + \ldots = \frac{x}{1-x}$ and $1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots = \frac{1}{1 - \frac{1}{x}} = \frac{x}{x-1}$. Adding these two equations

resulted in: $1 + x + \frac{1}{x} + x^2 + \frac{1}{x^2} + x^3 + \frac{1}{x^3} + \ldots = \frac{x}{1-x} + \frac{x}{x-1}$

$$= \frac{x}{1-x} + \frac{-x}{1-x}$$

$$= 0$$

However, if x is positive, the left side sums to a positive number! Where is the error in his reasoning?

11. Although each step of the following proof seems justified, the final result would cause Peano to rise from his grave! Where is the flaw?

$$(x+1)^2 = x^2 + 2x + 1 \longrightarrow (x+1)^2 - (2x+1) = x^2 \longrightarrow (x+1)^2 - (2x+1) - x(2x+1) = x^2 - x(2x+1) \longrightarrow$$

$$(x+1)^2 - (2x+1)(1+x) + \left(\frac{2x+1}{2}\right)^2 = x^2 - x(2x+1) + \left(\frac{2x+1}{2}\right)^2 \longrightarrow \left(x + 1 - \frac{2x+1}{2}\right)^2 = \left(x - \frac{2x+1}{2}\right)^2 \longrightarrow$$

$x + 1 - \frac{2x+1}{2} = x - \frac{2x+1}{2}$. Therefore, $x + 1 = x$!!

The Solutions

1. $\frac{10a+b}{10b+c}=\frac{a}{c} \longrightarrow 10ac+bc=10ab+ac \longrightarrow 9ac+bc=10ab \longrightarrow c=\frac{10ab}{9a+b}$.

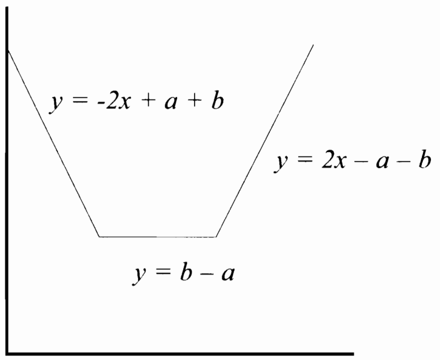
   Therefore, $9a+b$ must evenly divide $10ab$. This is always true when $a=b=c$ and in four other situations: $a=1, b=6, c=4; a=1, b=9, c=5; a=2, b=6, c=5; a=4, b=9, c=8$, producing

   $\frac{16}{64}=\frac{1}{4}$, $\mathbf{\frac{19}{95}=\frac{1}{5}}$, $\mathbf{\frac{26}{65}=\frac{2}{5}}$, $\frac{49}{98}=\frac{4}{8}$. (The last one is not acceptable because $\frac{4}{8}$ is not reduced.)

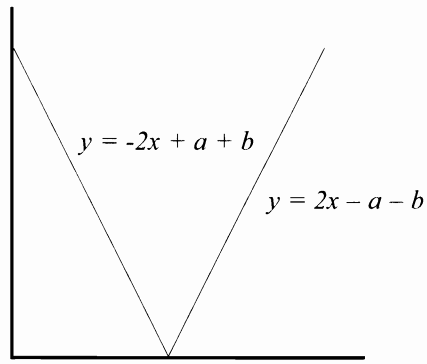
2. $a+(bc)=(a+b)(a+c) \longrightarrow a+(bc)=a^2+ac+ab+bc \longrightarrow a-a^2-ab-ac=0 \longrightarrow a(1-a-b-c)=0$

   $\therefore\ \boldsymbol{a=0 \text{ or } a+b+c=1}$.

3. With $a<b$, let $f(x)=|x-a|+|x-b|$ and $g(x)=|2x-a-b|$. The graphs below show that the functions are identical when $y>b-a$. Therefore, **c > b - a**



f(x)



g(x)

4. Ning's method of solution is correct. However, since the square root of a number is always positive, the greatest x can be is 16. Therefore, all values of x between 4 and 16, inclusively , solve the equation. (a must be between 0 and 16, inclusively.)

5. $\log(b^n)=n\log(b)$. Let $x=\log(b)$. Therefore, $nx=x^n \rightarrow 0=x^n-nx \rightarrow 0=x(x^{n-1}-n)$

   $\therefore\ x=0 \text{ or } x^{n-1}=n$. This implies log($b$) = 0 or $(\log b)^{n-1}=n$

   Case 1: log(b) = 0, implies **b = 1 and n is any positive integer**.

   Case 2 : **n = 1** implies $x^0=1$, which is true for any positive x. So, **if *n* = 1, *b* is any positive number** and if $\boldsymbol{n>1}$ then $\boldsymbol{b=10^{\sqrt[n-1]{n}}}$. (The exponent of 10 can be positive or negative if n is odd.)

6. Method 1: $\dfrac{a+\frac{b}{c}}{d+\frac{e}{f}} = \dfrac{\frac{ac+b}{c}}{\frac{df+e}{f}} = \dfrac{acf+bf}{cdf+ce}$ Method 2: $\dfrac{a+\frac{b}{c}}{d+\frac{e}{f}} = \dfrac{a+bf}{d+ce}$.

   Therefore, $\dfrac{acf+bf}{cdf+ce} = \dfrac{a+bf}{d+ce} \longrightarrow acdf + bdf + ac^2ef + bcef = acdf + ace + bcdf^2 + bcef \longrightarrow$

   $ac^2ef - ace = bcdf^2 - bdf \longrightarrow ace(cf-1) = bdf(cf-1) \rightarrow \mathbf{ace = bdf}$.

7. $\frac{a}{b} + \frac{c}{d} \rightarrow \frac{ad+bc}{bd}$. Therefore, $\frac{ad+bc}{bd} = \frac{a+c}{b+d}$. And so, $abd + bcd = abd + ad^2 + b^2c + bcd \longrightarrow$

   $\mathbf{ad^2 + b^2c = 0}$.

8. Start with a quadratic that is factorable: $(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$ and follow the algorithm:

   $acx^2 + (ad+bc)x + bd \rightarrow x^2 + (ad+bc)x + acbd \rightarrow (x+ad)(x+bc) \rightarrow (x+\frac{ad}{ac})(x+\frac{bc}{ac}) \rightarrow$

   $(x+\frac{d}{c})(x+\frac{b}{a}) \longrightarrow \mathbf{(cx+d)(ax+b)}$.

9. Board's equation: $x^2 - Ax + 10t + u = 0$; Sindhuja's equation: $x^2 + Ax - 10u - t = 0$ Let r_1 and r_2 be the roots of the first equation and r_2 and r_3 be the roots of the second. Then $r_1 + r_2 = A$ and $r_2 + r_3 = -A$, implying $\mathbf{r_1 + r_3 = -2r_2}$. Also $r_1 \cdot r_2 = 10t + u$ and $r_2 \cdot r_3 = -10u - t$, implying $r_1 \cdot r_2 + r_2 \cdot r_3 = 9t - 9u$. Therefore, $r_2(r_1 + r_3) = 9(t-u) \rightarrow r_2(-2r_2) = 9(t-u) \rightarrow -2r_2^2 = 9(t-u)$. Therefore, $\mathbf{r_2 = 3}$ and $t - u = -2$. Therefore, the possible values for the constant term of the original equation are 13, 24, 35, 46, 57, 68, and 79. But the only ones that are divisible by 3 are 24 and 57, implying r_1 is either 8 or 19. Therefore, the original equation was either $\mathbf{x^2 - 11x + 24 = 0}$ or $\mathbf{x^2 - 22x + 57 = 0}$.

10. $a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}$, provided $|r| < 1$. Only one of the two equations can be correct because either $|x| < 1$ or $\left|\frac{1}{x}\right| < 1$, but not both.

11. If $a^2 = b^2$, then $a = b$ or $a = -b$. Therefore, the penultimate line should read:

    $\left(x + 1 - \frac{2x+1}{2}\right)^2 = \left(x - \frac{2x+1}{2}\right)^2 \longrightarrow x + 1 - \frac{2x+1}{2} = x - \frac{2x+1}{2}$ **or** $x + 1 - \frac{2x+1}{2} = -x + \frac{2x+1}{2}$. The left solution implies $x = x + 1$, an impossibility; while the right solution implies $0 = 0$, affirming that the original equation is always true!

Mathematical Strings

The Definitions

A mathematical string is any ordered list of symbols, where a symbol can occur more than once. While strings occur in probability theory, topology, combinatorics, set theory, and other branches of mathematics, they are also very commonplace in today's world. Words, phone numbers, license plate codes, UPC symbols are all examples of strings. Using the letters from a set, such as $\{a, b, c\}$, this problem set will look at strings such as $aaaa$, $ababba$, and $bbccbccbb$. The length of a string, n, is equal to the number of symbols in the string.

The Problems

1a. Using the set of symbols, $\{a, b, c, d, e\}$, how many strings of length n can be made? Justify your answer.

1b. Using the set of symbols, $\{a, b, c, d, e\}$, how many palindromic strings of length n can be made? (A palindromic string is the same when read forwards or backwards, like *abba, ccc,* and *bacab.*) Justify your answer.

1c. Using the set of symbols, $\{a, b, c, d, e\}$, how many strings of length n $(n \geq 2)$ can be made that contain exactly 2 $a's$?

Using an ordered set of symbols, $\{a_1, a_2, a_3, ..., a_k\}$, where $a_1 < a_2 < a_3 < ... < a_k$, many nondecreasing strings of length n can be made. (A string is considered nondecreasing if for each pair of adjacent symbols in the string, the symbol on the left side of the pair is less than or equal to the symbol on the right.) For example, using an ordered set of symbols, $\{a, b\}$, where $a < b$, there are three nondecreasing strings of length two (aa, ab, bb) and four nondecreasing strings of length three (aaa, aab, abb, bbb). In general, using an ordered set of symbols, $\{a, b\}$, where $a < b$, the number of nondecreasing strings of length n that can be made is $n+1$.

2a. Using an ordered set of symbols, $\{a, b, c\}$, where $a < b < c$, how many nondecreasing strings of length n can be made? Justify your answer.

2b. Using an ordered set of symbols, $\{a, b, c, d\}$, where $a < b < c < d$, how many nondecreasing strings of length n can be made? Justify your answer.

2c. Using an ordered set of symbols, $\{a_1, a_2, a_3, ..., a_k\}$, where $a_1 < a_2 < a_3 < ... < a_k$, how many nondecreasing strings of length n can be made? Justify your answer.

The string abc has six substrings: a, b, c, ab, bc, and abc. Using the symbols in the set $\{a, b\}$, $abbaa$ is a string with four substrings of length two: ab, bb, ba, and aa. With a length of five, $aabba$ is an example of the longest string with no repeated substrings of length two and $aaabbbabaa$ is an example of the longest string with no repeated substrings of length three.

3a. Using the symbols in the set $\{a, b\}$, how long is the longest possible string with no repeated substrings of length four? Give an example of such a string.

3b. Using the symbols in the set $\{a, b, c\}$, give an example of the longest possible string with no repeated substrings of length two. How long is your example?

3c. Using the symbols in the set $\{a, b, c, d\}$, give an example of the longest possible string with no repeated substrings of length two. How long is your example?

3d. Using the set of distinct symbols $\{a_1, a_2, a_3, \ldots, a_k\}$, how long is the longest possible string with no repeated substrings of length m? Justify your answer.

In many games, such as tennis, volleyball, ping-pong, and ultimate frisbee, you must win by two points. For example, in a high school volleyball game, generally the first team to reach 15 points is the winner. However, the game can not end with a score of 15 to 14. Therefore, the score must have been tied at 14 all and "overtime" must be played. If a represents Team A winning a point following the 14-14 tie, and b represents Team B winning a point, a string of a's and b's could represent the "overtime" portion of the game. The strings bb and $abbaaa$ could be strings representing the overtime, while $abbaa$ and $abaaaa$ could not represent an overtime.

4a. List all the "overtime" strings of length 6. How many are there?

4b. Determine a formula for the number of "overtime" strings of length n, where n is any positive even integer.

4c. If two teams are evenly matched, what is the average or expected length of an overtime string? Show your work, justifying your answer.

A string consisting of two a's and three b's could represent the counting of ballots in an election where candidate A receives two votes and candidate B receives three votes. The ten different arrangements of these a's and b's ($aabbb$, $ababb$, $abbab$, $abbba$, $baabb$, $babab$, $babba$, $bbaab$, $bbaba$, $bbbaa$)) could represent the ten different ways of the counting the five ballots. Notice that in only two of the ten countings, namely $bbaba$ and $bbbaa$, candidate B is always ahead of candidate A at each step in the count. If $N(A)$, the number of votes for candidate A, equals 2 and $N(B)$, the number of votes for candidate B, equals 3, the probability that candidate B is always ahead in the count is $2/10$ or $1/5$.

5a. If $N(A) = 3$ and $N(B) = 4$, how many different arrangements of a's and b's are there?

5b. If $N(A) = 3$ and $N(B) = 4$, make a list of the strings where candidate B is always ahead of candidate A throughout the counting of the ballots.

5c. In general, if $N(A)$ represents the number of votes for candidate A and $N(B)$, the number of votes for candidate B, with $N(B) > N(A)$, what is the probability, in terms of $N(A)$ and $N(B)$, that candidate B will always be ahead of candidate A throughout the counting of the ballots? Justify your answer.

The Solutions

1a. 5^n There are five choices for each of the n spots.

1b. If n is even, $5^{\frac{n}{2}}$ and, if n is odd, $5^{\frac{n+1}{2}}$. There are five choices for only half of the spots.

1c. The two spots to place the a's can be selected in $\binom{n}{2}$ ways. There are 4 choices for each of the remaining $n-2$ spots. Therefore, there are $\binom{n}{2}(4^{n-2})$ strings with exactly two a's.

2a. $\dfrac{(n+1)(n+2)}{2}$

n	$k=3$	
1	3	- *a, b, c*
2	6	- *aa, ab, ac, bb, bc, cc*
3	10	- *aaa, aab, aac, abb, abc, bbb, bbc, acc, bcc, ccc*
4	15	- 1(3) + 3(2) + 6(1)

...The triangle numbers!

2b. $\dfrac{(n+1)(n+2)(n+3)}{6}$

n	$k=4$	
1	4	- *a, b, c, d*
2	10	- *aa, ab, ac, ad, bb, bc, bd, cc, cd, dd*
3	20	- *aaa, aab, aac, aad, abb, abc, abd, acc, acd, add, bbb, bbc, bbd, bcc, bcd, bdd, ccc, ccd, cdd, ddd*
4	35	- 1(10) + 2(6) + 3(3) + 4(1) (From n=3, <u>a</u> can preceed all strings beginning with <u>a</u>, both <u>a</u> and <u>b</u> can preceed all strings beginning with <u>b</u>, etc.)

...The tetrahedral numbers!!

2c. $\dfrac{\prod_{i=1}^{k-1}(n+i)}{(k-1)!}$

or

$\binom{n+k-1}{n}$

	k	
$\{a, b\}$	2	$n+1$
$\{a, b, c\}$	3	$\dfrac{(n+1)(n+2)}{2}$
$\{a, b, c, d\}$	4	$\dfrac{(n+1)(n+2)(n+3)}{6}$
$\{a, b, c, d, e\}$	5	$\dfrac{(n+1)(n+2)(n+3)(n+4)}{24}$

3a. ***aaaabbbbabbaababaa***

length **= 19**

3b. ***aabbaccbcc***

length **= 10**

3c. ***aabbccddacbdcadb***

length $= \mathbf{17}$

3d. $k^m + m - 1$ (There are k^m strings of length m using k different symbols (k choices for each of the m spots) and, with proper arrangement and staggered alignment, the letters in each column are identical. The first symbol in each row is used to form the string, along with the remaining $m - 1$ symbols of the last string. (See diagram at the right and discussion about this problem following the solution to problem 5.)

```
aaa     aaa          a
aab      aab          a
abb       abb          a
bbb        bbb          b
bba         bba          b
bab          bab          b
aba           aba          a
baa            baa          b  aa
```

4a. There are eight: ***ababaa***, ***ababbb***, ***abbaaa***, ***abbabb***, ***baabaa***, ***baabbb***, ***babaaa***, ***bababb*** .

4b. To form all the overtime strings of length n, tack an ***ab*** or ***ba*** onto the beginning of all the overtime strings of length $n-2$.

Therefore , $T_n = T_{n-2} * 2$, ***where*** $T_2 = 2$ or $t(n) = 2^{\frac{n}{2}}$ ***for all positive even*** n .

4c. If two teams are evenly matched half of the the overtimes will be represented by strings of length two, a fourth by strings of length four, an eighth by strings of length six, and so on. The <u>expected</u> length of a overtime period would be the sum of all possible overtimes, each multiplied by the probability that the overtime goes that long: $2(1/2) + 4(1/4) + 6(1/8) + 8(1/16) + 10(1/32) + 12(1/64) + \ldots = 1 + 1 + 3/4 + 1/2 + 5/16 + 6/32 + \ldots$

Separate this sum into some partial sums:

$= 1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \ldots$

$+ 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \ldots$

$+ 1/4 + 1/8 + 1/16 + 1/32 + \ldots$

$+ 1/8 + 1/16 + 1/32 + \ldots$

$+ 1/16 + 1/32 + \ldots$

$+ 1/32 + \ldots$

$+ \ldots$

Since each sum is an infinite geometric series, each has a finite sum:

$$= 1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \ldots = \frac{1}{1 - 1/2} = 2$$

$$+ 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \ldots = \frac{1/2}{1 - 1/2} = 1$$

$$+ 1/4 + 1/8 + 1/16 + 1/32 + \ldots = \frac{1/4}{1 - 1/2} = \frac{1}{2}$$

$$+ 1/8 + 1/16 + 1/32 + \ldots = \frac{1/8}{1 - 1/2} = \frac{1}{4}$$

$$+ 1/16 + 1/32 + \ldots = \frac{1/16}{1 - 1/2} = \frac{1}{8}$$

$$+ 1/32 + \ldots = \frac{1/32}{1 - 1/2} = \frac{1}{16}$$

$$+ \ldots$$

The sum of these finite sums again forms an infinite geometric series with a finite sum:

$$= 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \ldots = \frac{2}{1 - \tfrac{1}{2}} = \mathbf{4.}$$

5a. From the seven possible spots, select 3 spots to be represented by a vote for A. This can be done in $\binom{7}{3}$ ways.

Fill in the remaining four spots with votes for B. $\binom{7}{3} = \mathbf{35.}$

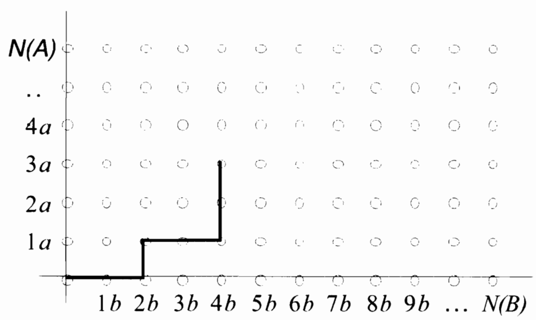
5b. There are five: ***bbbbaaa, bbbabaa, bbbaaba, bbababa, bbabbaa***

5c. Collect more data:

$N(A)$	$N(B)$	P
1	2	$\frac{1}{3}$
1	3	$\frac{2}{4} = \frac{1}{2}$
2	3	$\frac{2}{10} = \frac{1}{5}$
2	4	$\frac{5}{15} = \frac{2}{6}$
2	5	$\frac{9}{21} = \frac{3}{7}$
3	4	$\frac{5}{35} = \frac{1}{7}$
3	5	$\frac{14}{56} = \frac{2}{8}$

Notice that: $P = \dfrac{N(B) - N(A)}{N(B) + N(A)}$

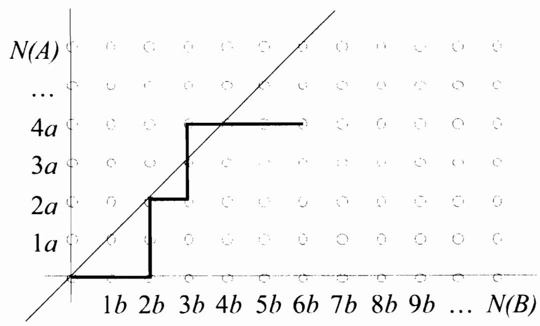
A string of $N(A)$ a's and $N(B)$ b's can be thought of as a "northeast path" on a lattice grid from (0,0) to ($N(B)$, $N(A)$), where a means "go up" and b means "go right". The path representing the string *bbabbaa* from part 5b is shown below:



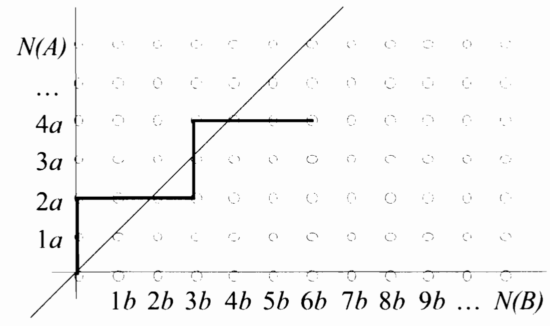
Since there are $\dfrac{(N(A) + N(B))!}{N(A)!N(B)!}$ arrangements of $N(A)$ a's and $N(B)$ b's, this also represents the number of "northeast paths" from (0, 0) to ($N(B)$, $N(A)$).

For candidate B to always be ahead of candidate A in the ballot counting, the sting of a's and b's must start with a b and $N(B)$ must be greater than $N(A)$. All the strings starting with a b can be represented by a path going to the right first. The number of such paths is equivalent to the number of paths going from ($1b$, 0) to ($N(B)$, $N(A)$). There are $\dfrac{(N(A) + (N(B) - 1))!}{N(A)!\,(N(B) - 1)!}$ such paths.

However, for candidate B to always be ahead of candidate A in the ballot counting, these paths cannot cross the diagonal line from (0, 0) to ($1b$, $1a$) to ($2b$, $2a$) to ($3b$, $3a$), etc. For each of these paths, starting at ($1b$, 0) and crossing the diagonal, there exists a dual path formed by taking the portion of the path from ($1b$, 0) to (mb,ma), the point where the "northeast path" first crosses this diagonal, and reflecting this portion across the diagonal. The following illustrations show an example of a path and its dual for strings of a's and b's:



bbaabaabbb



it's dual

These duals represent all the northeast paths from (0, 0) to (*N(B), N(A)*) going through ($1a$, 0). The number of such paths is equal to the number of paths from ($1a$, 0) to (*N(B), N(A)*) and there are $\frac{((N(A)-1)+N(B))!}{(N(A)-1)!N(B)!}$ of these paths.

Therefore, in a string of N(A) a's and N(B) b's representing ballot counting, the probability that candidate B will always be ahead of candidate A would be:
$$\frac{\frac{(N(A)+(N(B)-1))!}{N(A)!(N(B)-1)!}-\frac{((N(A)-1)+N(B))!}{(N(A)-1)!N(B)!}}{\frac{(N(A)+N(B))!}{N(A)!N(B)!}}=$$

$$\frac{(N(A)+N(B)-1)!N(B)-(N(A)+N(B)-1)!N(A)}{(N(A)+N(B))!}=\frac{(N(A)+N(B)-1)!(N(B)-N(A))}{(N(A)+N(B))!}=\frac{(N(A)+N(B)-1)!(N(B)-N(A))}{(N(A)+N(B))(N(A)+N(B)-1)!}=$$

$$\frac{N(B)-N(A)}{N(A)+N(B)}.$$

<u>Solution to #3 continued.</u>

Although many teams may have solved #3 using a "guess and check" method, the following methods may appear useful and instructive. Given a set of k different letters or symbols, the length of the longest string containing no substrings of length m is $k^m + m - 1$. This is because there are k^m strings of length m using the S different symbols. They overlap each other, with the initial letter of each substring being forming the consecutive letters of the desired string. However, the last substring has $m - 1$ more letters to be attached to the end of the desired string.

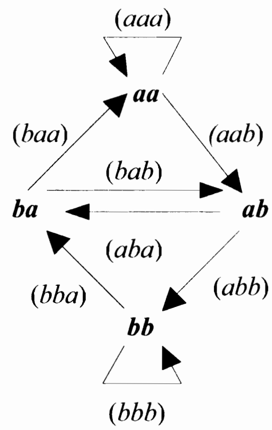
The longest string using $\{a, b\}$ with no repeated substrings of length 5:
aaaaababbbababaabaaabbabbaabbbbbaaaa

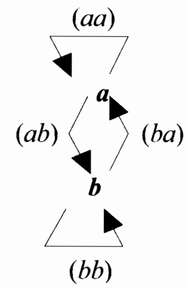
The 32 substrings of length 5 from $\{a, b\}$ showing how they overlap (Decimal equivalents of the binary numbers represented by the strings with $a = 0$ and $b = 1$):

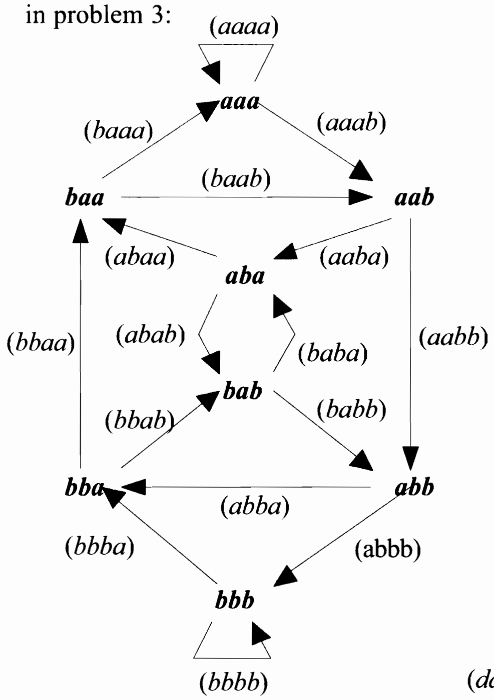
aaaaa	0
aaaab	1
aaaba	2
aabab	5
ababb	11
babbb	23
abbba	14
bbbab	29
bbaba	26
babab	21
ababa	10
babaa	20
abaab	9
baaba	18
aabaa	4
abaaa	8
baaab	17
aaabb	3
aabba	6
abbab	13
bbabb	27
babba	22
abbaa	12
bbaab	25
baabb	19
aabbb	7
abbbb	15
bbbbb	31
bbbba	30
bbbaa	28
bbaaa	24
baaaa	16

Therefore, the length of the longest possible string with no repeated substrings of length m is known, but how does one arrange all the substrings to form this string and is it always possible to do so? Euler circuits and graph theory give us the answer!

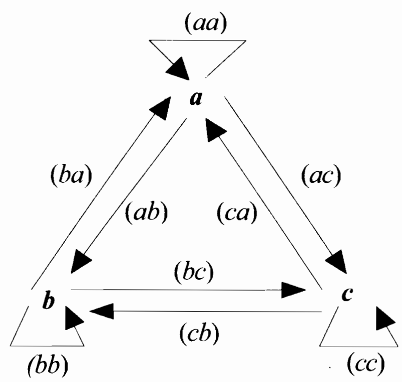
In problem 3, you were given the longest string of *a*'s and *b*'s with no repeated substrings of length 2 or 3. Here is a method from graph theory for finding the second one when you know the first one. Let the vertices of the graph be all the strings of *a*'s. and *b*'s of length 2. (There are four of them.) Vertex *ab* is connected with an arrow to vertices *ba* and *bb*, forming directed edges *aba* and *abb*, respectively. (These are the only substrings of length 3 that start with *ab* . Notice that they end with *ba* and *bb* .) Now do this to the other three vertices: *aa* is connected to *aa* and *ab*, forming directed edges *aaa* and *aab* ; *ba* is connected to *aa* and *ab*, forming directed edges *baa* and *bab*; *bb* is connected to *ba* and *bb*, forming directed edges *baa* and *bbb*. Since this is a connected graph and very vertex has an even degree of 4, Euler proved that this graph has an Euler circuit, i.e., starting at any vertex you can travel each of the edges without having to ever retrace your path. Here are the graphs for some of the other strings in problem 3:





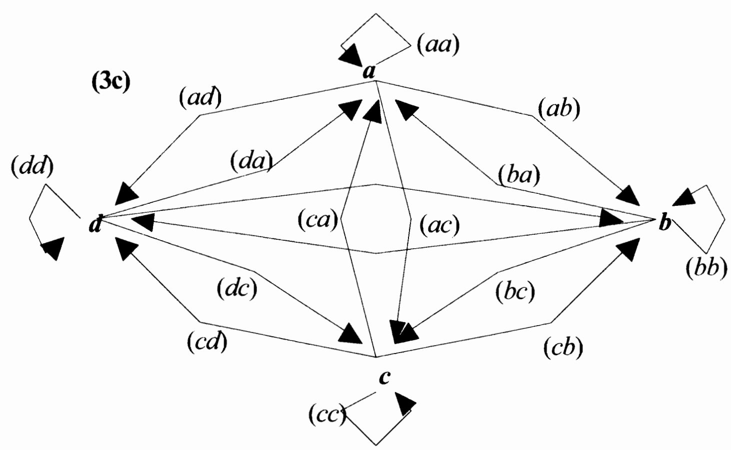


(3a)



(3b)

(3c)



ARML is like no other mathematics contest. After months of planning and preparations, tryouts and practice sessions, busloads of students stream onto three college campuses, turning sedate institutions into beehives of excitement and anticipation. New friendships are made; old ones are renewed. Superb mathematics students from across the country are drawn together by their love of mathematics, eager to measure their abilities against other talented students as well as against a collection of truly challenging non-routine problems. ARML is distinguished by the fact that it creates communities of mathematics students and teachers. ARML's format provides for a variety of problem-solving situations and the fine problems that distinguish ARML provoke and promote mathematical insight and inventiveness. This book contains problems and solutions from the 1995 to 2003 ARML contests.

In 1994 ARML introduced the ARML Power Contest, a competition by mail. Modeled on the Power Question, it asks teams of high school students to explore mathematical ideas through solutions to sequences of related problems. This book contains the problems and solutions from the 1994 to 2004 Power Contests.