ARML Competition 2025

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1 Team Round

Problem 1. Sunny and Moonbeam are at the two ends of a straight 1-mile walking trail. They begin walking toward each other along the trail at the same time. Sunny walks at 4 miles per hour, and Moonbeam walks at 3 miles per hour. They meet at point A and continue their respective walks. At point B, Sunny turns around and starts walking in the same direction as Moonbeam. They both reach the end of the trail at the same time. Compute the number of miles in AB.

Problem 2. A square is circumscribed about the graph of $|x|+|y| = 20$ and is inscribed in the graph of $|x|+|y| = 25$. Compute the area of this square.

Problem 3. Suppose that $\triangle ABC$ has $\tan A + \tan B = 2025$ and $\cot A + \cot B = 1976$. Given that $\sqrt{\tan C} = q\sqrt{r}$, where q is a rational number and r is a squarefree integer, compute r.

Problem 4. Consider the cubic polynomials $p(x) = x^3 + 7x^2 + 14x + 8$ and $q(x) = ax^3 + bx^2 + cx + d$. Suppose that the graphs of $y = p(x)$ and $y = q(x)$ are symmetric across the point $(4, 5)$. Compute the unique real solution to the equation $q(x) + 130 = 0$.

Problem 5. The sport of llama slabber requires four players from a team for each game. For any two games that a team plays in a season, they must use exactly two of the same players in both games, and no two players can play together in more than two games. Let G be the maximum number of games that a team can play in a season, and let P be the minimum number of players that a team needs in order to play G games in a season. Compute the ordered pair (G, P).

Problem 6. Let $\triangle ABC$ be isosceles with $AB = 1$. Suppose that there is exactly one point C' on $\overleftrightarrow{BC}$, distinct from B and C, such that $\triangle ABC'$ is also isosceles. Compute the sum of all distinct possible lengths AC.

Problem 7. Silas lines up 12 fair coins and flips each of them. Marnie then removes any coin that shows the same face as one or both of its neighbors. Compute the probability that at most one coin remains.

Problem 8. An unfair die has faces labeled 1 through 6 like a standard die, where the face k appears with probability p_k. Suppose that:

- Each of p_1, p_4, p_5, p_6 is at least $\frac{1}{7}$, and $p_2 = p_3 = \frac{1}{6}$.
- The expected value of a single roll matches that of a fair die.
- The expected value of the square of a single roll matches that of a fair die.

Compute the least possible value of p_1.

Problem 9. For each prime p, let U_p denote the set of ordered triples $(a, b, c) \in \{1, 2, \ldots, p\}^3$ for which

- there exists an integer x for which p divides $x^2 - a$,
- there exists an integer y for which p divides $y^3 - b$,
- there exists an integer z for which p divides $z^5 - c$, and
- p divides $a + b + c$.

Compute the least prime p for which $|U_p| > 2025$.

Problem 10. The trisectors of $\angle A$ in $\triangle ABC$ intersect the circumcircle of the triangle at points D and E so that the points A, B, D, E, and C lie on the circumcircle in this order. Given that $AB = 4$, $AD = 16$, and $AC = 14$, compute BC.

2 Team Round Answers

Answer 1. $\dfrac{2}{21}$

Answer 2. 1000

Answer 3. 494

Answer 4. 5

Answer 5. $(7, 7)$

Answer 6. $1 + \sqrt{2} + \sqrt{3}$

Answer 7. $\dfrac{329}{2048}$

Answer 8. $\dfrac{52}{315}$

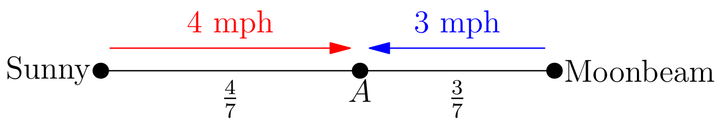
Answer 9. 83

Answer 10. $5\sqrt{11}$

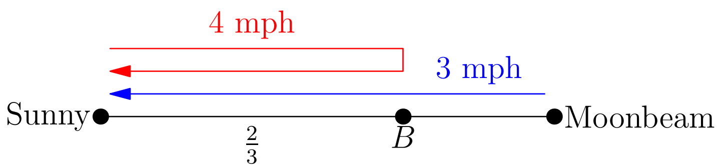
3 Team Round Solutions

Problem 1. Sunny and Moonbeam are at the two ends of a straight 1-mile walking trail. They begin walking toward each other along the trail at the same time. Sunny walks at 4 miles per hour, and Moonbeam walks at 3 miles per hour. They meet at point A and continue their respective walks. At point B, Sunny turns around and starts walking in the same direction as Moonbeam. They both reach the end of the trail at the same time. Compute the number of miles in AB.

Solution 1. The locations of points A and B can be independently computed as follows. Sunny and Moonbeam first meet after they have collectively walked 1 mile, and this occurs after $\frac{1}{4+3} = \frac{1}{7}$ hours. Hence point A is $4 \cdot \frac{1}{7} = \frac{4}{7}$ miles from Sunny's starting point.



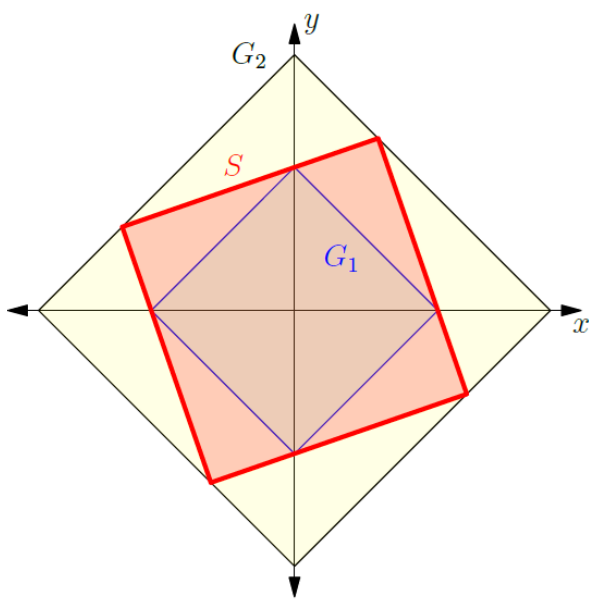
The second meeting occurs after Moonbeam has walked 1 mile, the length of the entire trail. Because Sunny walks at $\frac{4}{3}$ the rate of Moonbeam, it follows that Sunny walked a total of $4/3$ miles. Thus B is $2/3$ miles from Sunny's starting point.



Therefore the number of miles in AB is $|2/3 - 4/7| = \mathbf{2/21}$.

Problem 2. A square is circumscribed about the graph of $|x|+|y| = 20$ and is inscribed in the graph of $|x|+|y| = 25$. Compute the area of this square.

Solution 2. The graph G_1 of $|x| + |y| = 20$ is a square with side length $20\sqrt{2}$. The graph G_2 of $|x| + |y| = 25$ is a square with side length $25\sqrt{2}$. Let the inscribed square S in the problem statement have side length s.



Corresponding sides of G_1 and G_2 are parallel to each other. Hence the angles between a side of G_1 and the two sides of S touching it are congruent to the angles between a side of S and the two sides of G_2 touching it. It follows that the similarity ratio between G_1 and S equals that between S and G_2. This implies

$$\frac{25\sqrt{2}}{s} = \frac{s}{20\sqrt{2}},$$

so the desired area is

$$s^2 = 20\sqrt{2} \cdot 25\sqrt{2} = \mathbf{1000}.$$

Alternate Solution: As before, let the inscribed square S in the problem statement have side length s. Impose a coordinate system where one of the vertices of S is (a, b), with $a + b = 25$ and $a,\ b > 0$. Then the other three vertices are $(-b, a)$, $(-a, -b)$, and $(b, -a)$. The line passing through (a, b) and $(-b, a)$ must pass through $(0, 20)$, so it follows that

$$\frac{b-20}{a-0} = \frac{a-20}{-b-0} \implies a^2 + b^2 = 20(a + b) = 500.$$

The side length of the square is given by

$$s = \sqrt{(a + b)^2 + (b - a)^2} = \sqrt{2(a^2 + b^2)} = \sqrt{2 \cdot 500},$$

so $s^2 = \mathbf{1000}$. Note that solving for (a, b) from $a + b = 25$ and $a^2 + b^2 = 500$ yields the solutions $\{a, b\} = \left\{\frac{5}{2}(5 \pm \sqrt{15})\right\}$.

Problem 3. Suppose that $\triangle ABC$ has $\tan A + \tan B = 2025$ and $\cot A + \cot B = 1976$. Given that $\sqrt{\tan C} = q\sqrt{r}$, where q is a rational number and r is a squarefree integer, compute r.

Solution 3. Notice that

$$1976 = \frac{1}{\tan A} + \frac{1}{\tan B} = \frac{\tan B + \tan A}{\tan A \tan B} = \frac{2025}{\tan A \tan B},$$

which implies $\tan A \tan B = \frac{2025}{1976}$. Because $\tan C = \tan(180^\circ - (A + B)) = -\tan(A + B)$, it follows that

$$\tan C = -\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\frac{2025}{1 - 2025/1976} = \frac{2025}{49} \cdot 1976.$$

Thus $\sqrt{\tan C} = \frac{45}{7} \cdot \sqrt{1976} = \frac{90}{7}\sqrt{494}$. Because $494 = 2 \cdot 13 \cdot 19$ is squarefree, it follows that $r = \mathbf{494}$.

Alternate Solution: Rearranging the fractions

$$\frac{2025}{\tan A \tan B} = \frac{\tan A + \tan B}{\tan A \tan B} = \cot B + \cot A = 1976,$$

it follows that $\tan A \tan B = \frac{2025}{1976}$. For any triangle ABC, recall the identity

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

This leads to $2025 + \tan C = \frac{2025}{1976} \tan C$, thus $\tan C = \frac{2025 \cdot 1976}{49}$ and $\sqrt{\tan C} = \frac{45}{7}\sqrt{1976} = \frac{90}{7}\sqrt{494}$. Again, $r = \mathbf{494}$.

Problem 4. Consider the cubic polynomials $p(x) = x^3 + 7x^2 + 14x + 8$ and $q(x) = ax^3 + bx^2 + cx + d$. Suppose that the graphs of $y = p(x)$ and $y = q(x)$ are symmetric across the point $(4, 5)$. Compute the unique real solution to the equation $q(x) + 130 = 0$.

Solution 4. Instead of solving $q(x) = -130$, reflect the line $y = -130$ across the point $(4,5)$ to get the line $y = 140$. First consider the equation $p(x) = 140$, which is equivalent to $x^3 + 7x^2 + 14x - 132 = 0$, and note that $p(3) = 140$, which implies that $x = 3$ is a solution. This is the only real solution because $x^3 + 7x^2 + 14x - 132 = (x-3)(x^2 + 10x + 44)$, and for all real x, $x^2 + 10x + 44 = (x+5)^2 + 19 > 0$. Finally, reflecting the point $(3, 140)$ in $(4, 5)$ yields $(5, -130)$, so $x = \mathbf{5}$ is the desired solution.

Alternate Solution: Note that

$$p(x) = x^3 + 7x^2 + 14x + 8 = (x+1)(x^2 + 6x + 8) = (x+1)(x+2)(x+4).$$

Because the graphs of $y = p(x)$ and $y = q(x)$ are symmetric about the point $(4, 5)$, it follows that $p(4-x) + q(4+x) = 10$; that is, $(4, 5)$ is the midpoint of the segment with endpoints $\big(4-x, p(4-x)\big)$ and $\big(4+x, q(4+x)\big)$. Therefore

$$q(x+4) = 10 - p(4-x) = 10 - (5-x)(6-x)(8-x) = (x-5)(x-6)(x-8) + 10,$$

so $q(x) = (x-9)(x-10)(x-12) + 10$. Thus $(x-9)(x-10)(x-12) + 10 = -130$, which implies

$$(x-9)(x-10)(x-12) = -140 = (-4)(-5)(-7),$$

hence $x - 9 = -4$, so $x = \mathbf{5}$.

Problem 5. The sport of llama slabber requires four players from a team for each game. For any two games that a team plays in a season, they must use exactly two of the same players in both games, and no two players can play together in more than two games. Let G be the maximum number of games that a team can play in a season, and let P be the minimum number of players that a team needs in order to play G games in a season. Compute the ordered pair (G, P).

Solution 5. Denote the players by capital letters A, B, and so on. Call the players in the first game of the season A, B, C, and D. Each subsequent game after the first must use a distinct subset of those four players. Hence there are a maximum of $\binom{4}{2} = 6$ additional games, thus $G \leq 1 + 6 = 7$. To give an example that uses 7 games, note that the roster

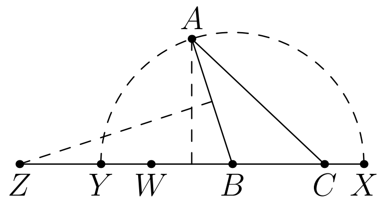
ABCD, ABEF, ACEG, ADFG, BCEG, BDFG, CDEF

is an example of seven games satisfying the requirements. Hence $G = 7$, and moreover, $P \leq 7$. It remains to verify that $P \geq 7$, i.e., to verify there is no construction of seven games using fewer than 7 players. In order for seven games to take place, after the first game, the remaining $\binom{4}{2} = 6$ games must have exactly two players among $\{\mathrm{A, B, C, D}\}$ and two new players not among $\{\mathrm{A, B, C, D}\}$. If there were only two new players (say, only E and F), then one cannot have them play in all six games. Hence a seventh player is necessary, so it follows that $P \geq 7$. In summary, $(G, P) = \mathbf{(7, 7)}$.

Problem 6. Let $\triangle ABC$ be isosceles with $AB = 1$. Suppose that there is exactly one point C' on $\overleftrightarrow{BC}$, distinct from B and C, such that $\triangle ABC'$ is also isosceles. Compute the sum of all distinct possible lengths AC.

Solution 6. Let ℓ be the line $\overleftrightarrow{BC}$ for shorthand. Call a point on ℓ a *potential location* for C' if two of AB, BC', and $C'A$ are equal. Call a potential location a *valid location* if, additionally, it is distinct from B and C. The problem statement requires exactly one valid location on ℓ. All possible potential locations are as follows:

- The potential locations with $AB = AC'$ are B and the reflection of B over the perpendicular from A to ℓ. Call this point W.
- The potential locations with $BC' = BA$ are the points on ℓ that are 1 unit away from B. Let X be the one on the same side of B as C, and Y be the one on the opposite side.
- The only potential location with $C'A = C'B$ is the intersection of the perpendicular bisector of $\overline{AB}$ with ℓ, if it exists. Call this point Z.



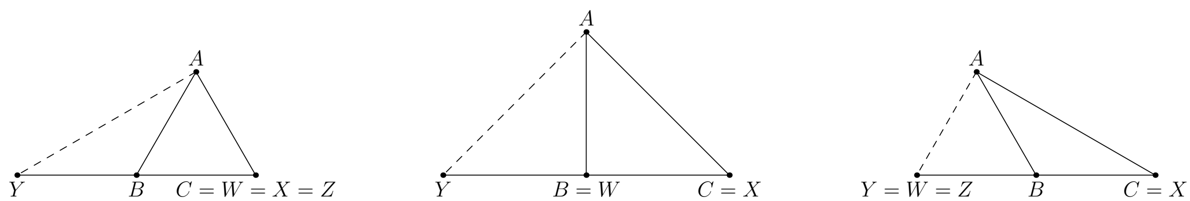
Note that Y is always valid, and X is always distinct from B and Y. Thus the condition of a unique valid location is equivalent to $W \in \{B, C, Y\}$, $X = C$, and either Z does not exist or $Z \in \{B, C, Y\}$. First, observe that the condition that $X = C$ is equivalent to $BC = 1$. The configuration is thus characterized by the angle θ between $\overline{AB}$ and $\overline{BC}$. Next, analyze the condition on W.

- If $W = B$, then $\overline{AB} \perp \ell$. This gives $\theta = 90°$.
- If $W = C$, then $\triangle ABC$ is equilateral. This gives $\theta = 60°$.
- If $W = Y$, then θ is obtuse and $BW = 1$. So $\triangle ABW$ is equilateral, which gives $\theta = 120°$.

Thus the condition on W is equivalent to $\theta \in \{60°, 90°, 120°\}$. Finally, analyze the condition on Z.

- If Z does not exist, then $\overline{AB} \perp \ell$. This gives $\theta = 90°$.
- Note that Z cannot equal B.
- If $Z = C$, then $\triangle ABC$ is equilateral. This gives $\theta = 60°$.
- If $Z = Y$, then θ is obtuse and $BZ = 1$. So $\triangle ABZ$ is equilateral, which gives $\theta = 120°$.

Thus the condition on Z is equivalent to $\theta \in \{60°, 90°, 120°\}$.



Thus the condition of a unique valid location is equivalent to $BC = 1$ and $m\angle ABC \in \{60°, 90°, 120°\}$. This corresponds to AC being 1, $\sqrt{2}$, and $\sqrt{3}$, respectively. Thus the sum of all possible distances AC is $\mathbf{1+\sqrt{2}+\sqrt{3}}$.

Problem 7. Silas lines up 12 fair coins and flips each of them. Marnie then removes any coin that shows the same face as one or both of its neighbors. Compute the probability that at most one coin remains.

Solution 7. Assume without loss of generality that the first flip is heads. Then denote each subsequent flip by an S if it is the same as the previous flip and a D if it is different. Appending an S to each end of the resulting sequence, there will be at most one coin remaining if there is at most one instance of two consecutive Ds in the sequence. For a set of n coins, let a_n be the number of such sequences (of length $n+1$) that contain exactly one instance of two consecutive Ds, and b_n be the number of such sequences (of length $n+1$) that contain no two consecutive Ds. Then note that $a_2 = 0$ and $b_2 = 1$. For $n > 2$, it follows that

$$b_n = b_{n-2} + b_{n-3} + \cdots + b_2 + 1$$

because a sequence of length n can be created from a sequence of length $n-k$ ($k \geq 2$), followed by $k-1$ Ds and an S, or from the sequence SDD...DS. Similarly, $a_n = b_{n-1} + a_{n-2} + a_{n-3} + \cdots + a_2 + 1$, because a sequence that has two consecutive Ds can be created in the same way as above, or by appending one more D to a sequence that does not have two consecutive Ds. Use this recursive formula to compute a_n and b_n for $n = 3, 4, \ldots, 12$, ending with $a_{12} = 240$ and $b_{12} = 89$. The number of possible sequences is 2^{11}, so the probability is $\mathbf{\frac{329}{2048}}$.

Alternate Solution: Each of the 2^{12} sequences of H and T corresponds to a way to write 12 as a sum of the repeat-lengths of Hs or Ts. For example, HHHHTTTTHHTH becomes the ordered partition $12 = 4+4+2+1+1$. Moreover, each ordered partition corresponds to two sequences of coins. Begin by computing the number of ordered partitions of n that have at most one 1. If there are no 1s, find the number of solutions to $12 = x_1 + x_2 + \cdots + x_k$, where $x_i \geq 2$. For a fixed k $(1 \leq k \leq 6)$, this is the number of nonnegative integer solutions to $12 - 2k = y_1 + y_2 + \cdots + y_k$, which, by the sticks-and-stones method, equals $\binom{11-k}{k-1}$. Summing from $k = 1, \ldots, 6$ gives

$$\sum_{k=1}^{6} \binom{11-k}{k-1} = \binom{10}{0} + \binom{9}{1} + \binom{8}{2} + \binom{7}{3} + \binom{6}{4} + \binom{5}{5} = 89.$$

(In fact, if 11 is replaced by the odd number $m \geq 1$, this can be shown to equal the m^{th} Fibonacci number, F_m. In this case, $F_{11} = 89$.) If there is one 1, find the number of solutions to $12 = x_1 + x_2 + \cdots + x_k$, where some $x_t = 1$ and $x_i \geq 2$ for $i \neq t$. For fixed k, there are k ways of selecting t; then again, applying the sticks-and-stones method gives $\binom{11-k}{k-2}$ solutions. Hence the total in this case is

$$\sum_{k=2}^{6} k\binom{11-k}{k-2} = 240.$$

In total, there are $89 + 240 = 329$ ordered partitions in which at most a single element is 1. Thus the answer is $\frac{2 \cdot 329}{2^{12}} = \mathbf{\frac{329}{2048}}$.

Problem 8. An unfair die has faces labeled 1 through 6 like a standard die, where the face k appears with probability p_k. Suppose that:

- Each of p_1, p_4, p_5, p_6 is at least $\frac{1}{7}$, and $p_2 = p_3 = \frac{1}{6}$.
- The expected value of a single roll matches that of a fair die.
- The expected value of the square of a single roll matches that of a fair die.

Compute the least possible value of p_1.

Solution 8. Let $p_k = \frac{1}{6} + t_k$ for $k \in \{1, 4, 5, 6\}$. Then the given conditions imply

$$\begin{aligned} t_1 + t_4 + t_5 + t_6 &= 0 \\ t_1 + 4t_4 + 5t_5 + 6t_6 &= 0 \\ t_1 + 16t_4 + 25t_5 + 36t_6 &= 0, \end{aligned}$$

or equivalently,

$$\begin{aligned} t_4 + t_5 + t_6 &= -t_1 \\ 4t_4 + 5t_5 + 6t_6 &= -t_1 \\ 16t_4 + 25t_5 + 36t_6 &= -t_1. \end{aligned}$$

Solving for t_4, t_5, t_6 yields

$$\begin{aligned} t_4 &= -10t_1 \\ t_5 &= 15t_1 \\ t_6 &= -6t_1. \end{aligned}$$

The first bulleted condition from the problem implies that $\min(t_4, t_5, t_6) \geq \frac{1}{7} - \frac{1}{6} = -\frac{1}{42}$. In other words, the range of possible values of t_1 is exactly

$$-\frac{1}{42} \cdot \frac{1}{15} \leq t_1 \leq \frac{1}{42} \cdot \frac{1}{10}.$$

The least possible value of p_1 is therefore

$$p_1 = \frac{1}{6} - \frac{1}{42} \cdot \frac{1}{15} = \mathbf{\frac{52}{315}}.$$

Problem 9. For each prime p, let U_p denote the set of ordered triples $(a, b, c) \in \{1, 2, \ldots, p\}^3$ for which

- there exists an integer x for which p divides $x^2 - a$,
- there exists an integer y for which p divides $y^3 - b$,
- there exists an integer z for which p divides $z^5 - c$, and
- p divides $a + b + c$.

Compute the least prime p for which $|U_p| > 2025$.

Solution 9. Assume $p > 5$ is a prime, and work with a, b, c modulo p. The solution proceeds in three steps.

Claim: $|U_p| \leq p\left(\frac{p+1}{2}\right)$.

To prove the claim, note that a and b uniquely determine c. Further note that there are exactly $\frac{p+1}{2}$ choices of a, and there are at most p choices of b. This implies the result. Consequently $|U_p| \geq 2025$ would imply $p(p+1) > 2025 \cdot 2$, meaning $p \geq 67$.

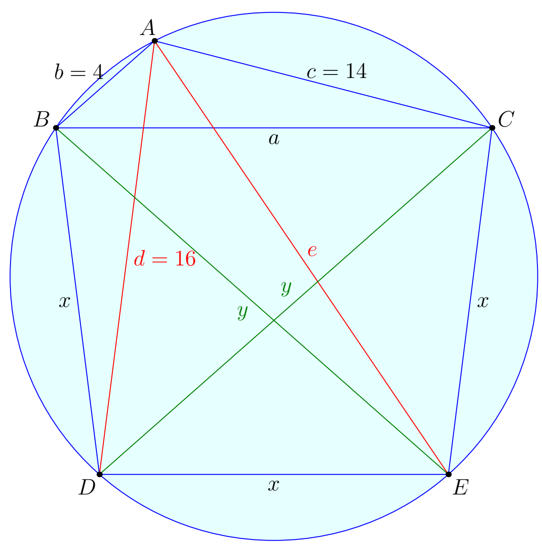
Claim: If $p \equiv 1 \pmod{h}$ for $h \in \{3, 5\}$, then $|U_p| < \left(\frac{p+1}{2}\right) \cdot \left(\frac{p-1}{h} + 1\right)$.

To prove the claim, consider that if $h = 3$, then there are at most $\frac{p-1}{3} + 1$ cubes modulo p, thus at most $\frac{p-1}{3} + 1$ values of b are allowed; then the value of a uniquely determines c. The argument is identical for $h = 5$, with the roles of b and c swapped.

Thus the primes $p = 67$, $p = 71$, $p = 73$, and $p = 79$ can be ruled out because in these cases, $p - 1$ has factors of 3, 5, 3, and 3, respectively. On the other hand, note that U_{83} has in fact exactly $83\left(\frac{83+1}{2}\right) = 3486 > 2025$ triples because every residue modulo 83 is both a perfect cube and a perfect fifth power, so the answer is **83**.

Problem 10. The trisectors of $\angle A$ in $\triangle ABC$ intersect the circumcircle of the triangle at points D and E so that the points A, B, D, E, and C lie on the circumcircle in this order. Given that $AB = 4$, $AD = 16$, and $AC = 14$, compute BC.

Solution 10. Let $BC = a$, $AB = b$, $AC = c$, $AD = d$, $AE = e$, $BD = DE = EC = x$, and $BE = CD = y$.



Using Ptolemy's Theorem in quadrilaterals $ABDE$ and $ACED$, it follows that $(b+e)x = dy$ and $(c+d)x = ey$. Therefore

$$\frac{y}{x} = \frac{b+e}{d} = \frac{c+d}{e} \implies \frac{4+e}{16} = \frac{30}{e} \implies e = 20.$$

Applying the Law of Cosines in triangles ABD and ADE yields

$$\cos\left(\frac{\angle BAC}{3}\right) = \frac{b^2+d^2-x^2}{2bd} = \frac{d^2+e^2-x^2}{2de}.$$

Hence $b(d^2+e^2-x^2) = e(b^2+d^2-x^2)$, so

$$x^2 = \frac{e(b^2+d^2)-b(d^2+e^2)}{e-b} = \frac{20(4^2+16^2)-4(16^2+20^2)}{20-4} = 176.$$

Therefore $x = 4\sqrt{11}$, and because $\frac{y}{x} = \frac{3}{2}$, it follows that $y = 6\sqrt{11}$. Using Ptolemy's Theorem in quadrilateral $BDEC$, it follows that $ax + x^2 = y^2$, so $a = \mathbf{5\sqrt{11}}$.

Alternate Solution: Let $\angle BAC = 3\theta < 180°$. Because triangles ABC, ABD, and ACD have the same circumcircle (of radius $2R$), by the Extended Law of Sines, it follows that $BC = 2R\sin 3\theta$, $BD = 2R\sin\theta$, and $CD = 2R\sin 2\theta$. Using Ptolemy's Theorem in quadrilateral $ABDC$,, it follows that $4\cdot CD + 14\cdot BD = 16\cdot BC$. Thus

$$8\sin 3\theta = 2\sin 2\theta + 7\sin\theta.$$

Using the triple and double angle identities for the sin function, it follows that

$$\begin{aligned} 8(3\sin\theta - 4\sin^3\theta) &= 2\sin\theta\cos\theta + 7\sin\theta \\ \iff 17 - 32\sin^2\theta &= 4\cos\theta \\ \iff 17 - 32(1-\cos^2\theta) &= 4\cos\theta \\ \iff 0 &= 32\cos^2\theta - 4\cos\theta - 15 \\ &= (8\cos\theta + 5)(4\cos\theta - 3). \end{aligned}$$

Hence $\cos\theta = \frac{3}{4}$ or $\cos\theta = -\frac{5}{8}$. However, because $\theta < 60°$, it follows that $\cos\theta = \frac{3}{4}$. Applying the Law of Cosines in triangle BAC yields

$$\begin{aligned} BC^2 &= 4^2 + 14^2 - 2\cdot 4\cdot 14\cdot\cos 3\theta \\ &= 212 - 112\left(4\cdot\left(\frac{3}{4}\right)^3 - 3\cdot\frac{3}{4}\right) \\ &= 275, \end{aligned}$$

so $BC = \mathbf{5\sqrt{11}}$.

4 Power Round 2025: The Bases are Loaded!

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

Define a $\{2,3\}$-*integer* to be a positive integer of the form $2^i 3^j$, where i and j are nonnegative integers. For positive integers n and r, a *double-base representation of n with length r* is an ordered r-tuple $(n_1, n_2, \ldots, n_r)$ such that $n = n_1 + \cdots + n_r$ and each n_k is a $\{2,3\}$-integer such that $n_1 < n_2 < \cdots < n_r$. The *length* of this double-base representation is r, and the n_k's are the *parts* of the double-base representation. For example, 8 has exactly three double-base representations: (8), $(2,6)$, and $(1,3,4)$, with lengths 1, 2, and 3, respectively. The notion of a double-base representation can be extended to $n = 0$: in this case, $r = 0$ (i.e., there are no $\{2,3\}$-integers that sum to 0), and the unique *empty double-base representation of* 0 is simply written as $()$.

A *strictly-chained double-base representation* of n is a double-base representation $(n_1, n_2, \ldots, n_r)$ of n such that n_k divides n_{k+1} for all $1 \le k < r$. For example, 8 has exactly two strictly-chained double-base representations: (8) and $(2,6)$ (note that $(1,3,4)$ is *not* a strictly-chained double-base representation of 8). Note that there is only one strictly-chained double-base representation of 0, namely $()$.

1. a. List two double-base representations of 24. [2 pts]

b. List **all** double-base representations of 15. [3 pts]

2. a. Compute the least positive integer n for which n has three different double-base representations with length exactly 3. [2 pts]

b. Find the number of distinct double-base representations of 2025 with length 2. [2 pts]

3. Show that every positive integer n has a double-base representation with length at most $\lfloor \log_3(n) \rfloor + 1$. [2 pts]

4. (a) Suppose that $n \ge 1$ and $n = n_1 + n_2 + \cdots + n_r$, where each n_k is a $\{2,3\}$-integer (not necessarily distinct). Show that n has a double-base representation with length at most r. [2 pts]

(b) Suppose that $n \ge 1$ has a double-base representation with length ℓ. Prove that n^2 has a double-base representation with length at most $\binom{\ell+1}{2}$. [3 pts]

5. A *signed double-base representation of n with length r* is an r-tuple $(n_1, \ldots, n_r)$ such that $n = n_1 + \cdots + n_r$, where $n_1 < n_2 < \cdots < n_r$ and $|n_k|$ is a $\{2,3\}$-integer for all $1 \le k \le r$. For example, $(-8, -2, 8, 9)$ is a signed double-base representation of 7 with length 4. Prove that a nonzero integer n with a signed double-base representation of length r also has a signed double-base representation of length $r+1$. [3 pts]

6. (a) List all strictly-chained double-base representations of 46 in which one of the parts is 1. [2 pts]

(b) List all strictly-chained double-base representations of 45. [2 pts]

(c) Let $p(n)$ denote the number of strictly-chained double-base representations of n, and let $p_1(n)$ denote the number of strictly-chained double-base representations of n that contain a 1 (i.e., $n_1 = 1$). Show that $p(n) = p_1(n) + p_1(n+1)$ for all positive integers n. [2 pts]

7. For each nonnegative integer n, let $f(n)$ be the number of double-base representations of n (note that $f(0) = 1$ because 0 has a unique empty double-base representation). In all parts below, assume that n is a positive integer.

(a) Show that there are exactly $f(n-1)$ double-base representations of n in which not all the parts are multiples of 3. [3 pts]

(b) Prove that if n is not divisible by 3, then $f(n) = f(n-1)$. [1 pt]

(c) Suppose that n is a multiple of 3. Show that $f(n) = f(n-1) + f\left(\frac{n}{3}\right)$. [3 pts]

The next two problems use the notation from Problem 6(c): $p(n)$ is the number of strictly-chained double-base representations of n. Note that $p(0) = 1$ because 0 has a unique empty strictly-chained double-base representation.

8. Determine all ordered pairs of integers (k, z) with $0 \leq z < k$ for which the following claim is true: if n is a positive integer such that $n - z$ is a multiple of k, then there are exactly $p\left(\frac{n-z}{k}\right)$ strictly-chained double-base representations of n in which every part is either z or a multiple of k. [4 pts]

9. The goal of this problem is to demonstrate that $p(n)$ satisfies a recurrence relation. In all parts below, assume that n is a positive integer.

(a) Show that $p(6n+1) = p(3n) + p(2n) - p(n)$. [3 pts]

(b) Suppose that n is a multiple of 3. Show that $p(n) = p(n-1) + p\left(\frac{n}{3}\right)$. [3 pts]

(c) Suppose that n leaves a remainder of 2 when divided by 3. Prove that $p(n) = p\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$. [3 pts]

(d) Prove that $p(3n+1) = p(n) + p\left(3 \cdot \left\lfloor \frac{n+1}{2} \right\rfloor - 1\right)$. [3 pts]

10. Find the number of strictly-chained double-base representations of 2025. [2 pts]

5 Power Round Solutions

1. (a) One such representation is $(4, 8, 12)$ because $24 = 4+8+12 = 2^2+2^3+(2^2 \cdot 3)$. Another such representation is $(8, 16)$ because $24 = 8+16 = 2^3+2^4$. The complete list of double-base representations of 24 is as follows:

- Representations with length 1: (24)
- Representations with length 2: $(6, 18)$, $(8, 16)$
- Representations with length 3: $(2, 4, 18)$, $(2, 6, 16)$, $(3, 9, 12)$, $(4, 8, 12)$
- Representations with length 4: $(1, 2, 3, 18)$, $(1, 2, 9, 12)$, $(1, 3, 4, 16)$, $(1, 3, 8, 12)$, $(1, 6, 8, 9)$, $(2, 4, 6, 12)$, $(3, 4, 8, 9)$
- Representations with length 5: $(1, 2, 3, 6, 12)$, $(1, 2, 4, 8, 9)$, $(2, 3, 4, 6, 9)$
- Representations with length 6: $(1, 2, 3, 4, 6, 8)$

(b) The complete list of the nine double-base representations of 15 is:

- Representations with length 2: $(3, 12)$, $(6, 9)$
- Representations with length 3: $(1, 2, 12)$, $(1, 6, 8)$, $(2, 4, 9)$, $(3, 4, 8)$
- Representations with length 4: $(1, 2, 3, 9)$, $(1, 2, 4, 8)$, $(2, 3, 4, 6)$

2. (a) The answer is **11**. Notice that $11 = 1 + 2 + 8 = 1 + 2 + 2^3$, $11 = 1 + 4 + 6 = 1 + 2^2 + 2 \cdot 3$, and $11 = 2 + 3 + 6 = 2 + 3 + 2 \cdot 3$. It can be confirmed that no positive integer less than 11 has three different double-base representations with length 3.

(b) The answer is **2**. Because 2025 is odd, it follows that one number in the double-base representation is a power of 3. Thus it follows that the maximum possible number of double-base representations of 2025 is 7 because $3^6 = 729 < 2025 < 2187 = 3^7$. Note that $2025-1 = 2024 = 2^3 \cdot 11 \cdot 23$, $2025-3 = 2022 = 2 \cdot 3 \cdot 337$, $2025 - 9 = 2016 = 2^5 \cdot 3^2 \cdot 7$, $2025 - 27 = 1998 = 2 \cdot 3^3 \cdot 37$, $2025 - 81 = 1944 = 2^3 \cdot 3^5$, $2025 - 243 = 1782 = 2 \cdot 3^4 \cdot 11$, and $2025 - 729 = 1296 = 2^4 \cdot 3^4$. Of these, only 1944 and 1296 are $\{2, 3\}$-integers. Thus the two double-base representations of 2025 are $(81, 1944)$ and $(729, 1296)$.

3. Let 3^r be the greatest power of 3 that is less than or equal to n. Note that the ternary (base-three) representation of n is of the form

$$n = a_r \cdot 3^r + a_{r-1} \cdot 3^{r-1} + \cdots + a_0 \cdot 3^0,$$

where $a_k \in \{0, 1, 2\}$ for all $0 \le k \le r$. Note that $z \cdot 3^x$ is a $\{2, 3\}$-integer for integers $x \ge 0$ and $z \in \{1, 2\}$. It follows that n is the sum of at most $r+1$ distinct $\{2, 3\}$-integers, and therefore n has a double-base representation with length at most $r+1$. However, note that $r = \lfloor \log_3(n) \rfloor$, so n has a double-base representation with length at most $\lfloor \log_3(n) \rfloor + 1$, as desired.

4. (a) Suppose that the n_k are distinct. Without loss of generality, assume $n_1 < n_2 < \cdots < n_r$. In this case, $(n_1, n_2, \ldots, n_r)$ is a double-base representation with length r.

Now suppose that some of the n_k are not distinct. Without loss of generality, assume $n_i = n_j$. Then the number $n_i + n_j = 2n_i$ is a $\{2, 3\}$-integer. Replacing n_i and n_j with $2n_i$, it follows that n can be written as the sum of $r - 1$ (not necessarily distinct) $\{2, 3\}$-integers. If these $\{2, 3\}$-integers are distinct, then the representation is double-base (after reordering the parts if needed), and the result follows. If not, repeat the process until the n_k are all distinct (again, reordering the parts if needed).

(b) Let $(n_1, \ldots, n_\ell)$ be a double-base representation of n with length ℓ. Then $n = n_1 + n_2 + \cdots + n_\ell$. Note that

$$n^2 = \sum_{i=1}^{\ell} n_i^2 + \sum_{1 \le i < j \le \ell} 2n_i n_j.$$

Because each n_k is a $\{2, 3\}$-integer, it follows that n_i^2 and $2n_i n_j$ are $\{2, 3\}$-integers for all $1 \le i, j \le \ell$. In other words, n^2 is the sum of $\ell + \binom{\ell}{2} = \binom{\ell+1}{2}$ numbers, each of which is a $\{2, 3\}$-integer. Using part (a), n^2 has a double-base representation with length at most $\binom{\ell+1}{2}$, as desired.

5. Let $(n_1, n_2, \ldots, n_r)$ be a signed double-base representation of n with length r. Then $n = n_1 + \cdots + n_r$, with $|n_k| = 2^{x_k} 3^{y_k}$ for all $1 \le k \le r$. Let n_i be a $\{2, 3\}$ integer such that $x_i + y_i$ is maximal. Then note that

$$|n_i| = 3|n_i| - 2|n_i| = 2^{x_i} 3^{y_i+1} - 2^{x_i+1} 3^{y_i},$$

so replacing n_i with $3n_i$ and $-2n_i$ does not change the sum on the right side of $n = n_1 + \cdots + n_r$ (and in particular, the sign of n_i does not matter). However, note that $3n_i$ and $-2n_i$ do *not* appear in the initial signed doubled-base representation of n. This is because the sum of the exponents in the prime factorizations of $3n_i$ and $-2n_i$ are both $x_i + y_i + 1 > x_i + y_i$. Hence replacing n_i with $3n_i$ and $-2n_i$ (and reordering the parts) yields a signed double-base representation of n with length $r + 1$, as desired.

6. (a) The strictly-chained double-base representations of 46 in which one of the parts is 1 are $(1, 3, 6, 12, 24)$, $(1, 3, 6, 36)$, and $(1, 9, 36)$.

(b) The strictly-chained double-base representations of 45 are $(1, 2, 6, 12, 24)$, $(1, 2, 6, 36)$, $(1, 4, 8, 32)$, $(3, 6, 12, 24)$, $(3, 6, 36)$, $(9, 36)$.

(c) Because there are $p_1(n)$ strictly-chained double-base representations of n that contain a 1, it suffices to show that there are $p_1(n + 1)$ strictly-chained double-base representations of n that do *not* contain a 1.

Given a strictly-chained double-base representation $(n_1, \ldots, n_r)$ of n that does not contain a 1, note that $(1, n_1, n_2, \ldots, n_r)$ is a strictly-chained double-base representation of $n + 1$ that contains a 1. Similarly, given a strictly-chained double-base representation $(1, n_1, \ldots, n_r)$ of $n + 1$ that contains a 1, note that $(n_1, \ldots, n_r)$ is a strictly-chained double-base representation of n that does not contain a 1.

Pairing up the strictly-chained double-base representations as described above yields a one-to-one correspondence between the strictly-chained double-base representations of n that do not contain a 1 and the strictly-chained double-base representations of $n + 1$ that contain a 1. Therefore there are $p_1(n + 1)$ strictly-chained double-base representations of n that do not contain a 1, hence $p(n) = p_1(n) + p_1(n + 1)$, as desired.

7. (a) Let $\mathcal{P}_n$ be the set of double-base representations $(n_1, \ldots, n_r)$ of n. In particular, $f(n) = |\mathcal{P}_n|$. As mentioned, $f(0) = 1$ because $\mathcal{P}_0 = \{()\}$. It suffices to find a bijection between the two sets

$$S = \big\{(n_1, n_2, \ldots, n_r) \in \mathcal{P}_n : 3 \nmid n_j \text{ for some } 1 \leq j \leq r\big\} \quad \text{and} \quad T = \mathcal{P}_{n-1}.$$

Consider some $(n_1, \ldots, n_r) \in S$. In other words, $n = n_1 + \cdots + n_r$ such that n_j is not a multiple of 3 for some $1 \leq j \leq r$. Let $I = \{i : 1 \leq i \leq r \text{ and } 3 \nmid n_i\}$, and note I is nonempty. Furthermore, n_i is a power of 2 for all $i \in I$, and thus $\sum\limits_{i \in I} n_i$ is the binary representation of $n - \sum\limits_{i \notin I} n_i \geq 1$. Let $x_1 + x_2 + \cdots + x_m$ be the binary representation of $n - 1 - \sum\limits_{i \notin I} n_i$.

Note that the equality $n - 1 = x_1 + \cdots + x_m + \sum\limits_{i \notin I} n_i$ (where one may exclude the x_j's if $x_1 + \cdots + x_m$ is the binary representation of 0) has the following properties:

- x_i is a power of 2 for each $1 \leq i \leq m$.
- n_i is a $\{2, 3\}$-integer that is a multiple of 3 for all $i \notin I$.
- The right side has $m + r - |I|$ distinct terms; that is, the numbers $x_1, \ldots, x_m$ and n_i for all $i \notin I$ are all pairwise distinct.

Let $y_1 < y_2 < \cdots < y_{m+r-|I|}$ such that $\{y_1, \ldots, y_{m+r-|I|}\} = \{x_1, \ldots, x_m\} \cup \{n_i : i \notin I\}$. Then it follows that $\big(y_1, \ldots, y_{m+r-|I|}\big)$ is a double-base representation for $n - 1$ because each y_i is a $\{2, 3\}$-integer such that $y_i < y_{i+1}$ for all $1 \leq i \leq m + r - |I|$. In particular, $\big(y_1, \ldots, y_{m+r-|I|}\big) \in T$.

Furthermore, this mapping is invertible. Consider $(n_1, \ldots, n_r) \in T$; then $(n_1, \ldots, n_r)$ is a double-base representation of $n - 1$. Let $J = \{j : 1 \leq j \leq r \text{ and } 3 \nmid n_j\}$ (note J could possibly be empty). Again, $\sum\limits_{j \in J} n_j$ is the binary representation of $n - 1 - \sum\limits_{j \notin J} n_j \geq 0$. Let $w_1 + w_2 + \cdots + w_{m'}$ be the binary representation of $n - \sum\limits_{j \notin J} n_j$. This is a nonempty binary representation because $n - \sum\limits_{j \in J} n_j \geq 1$.

Let $z_1 < z_2 < \cdots < z_{m'+r-|J|}$ such that $\{z_1, \ldots, z_{m'+r-|J|}\} = \{w_1, \ldots, w_{m'}\} \cup \{n_j : j \notin J\}$. Using similar logic as above, it follows that $\big(z_1, \ldots, z_{m'+r-|J|}\big)$ is a double-base representation for n. Furthermore, $3 \nmid z_i$ for some $1 \leq i \leq m' + r - |J|$ (in particular, this is true for each z_i equal to one of the w_j's), and therefore $\big(z_1, \ldots, z_{m'+r-|J|}\big) \in S$.

Note that these two mappings are inverses of each other. This bijection implies $|S| = |T|$, hence there are exactly $f(n - 1)$ double-base representations of n in which not all parts are multiples of 3, as desired.

(b) Note that if n is not a multiple of 3, then any double-base representation of n (and there are $f(n)$ such representations) has at least one part that is not a multiple of 3. From part (a), it follows that there are $f(n-1)$ double-base representations of n in which not all parts are multiples of 3, so $f(n) = f(n-1)$, as desired.

(c) From part (b), there are $f(n-1)$ double-base representations of n such that not all the parts are multiples of 3. It therefore suffices to show there are $f\left(\frac{n}{3}\right)$ double-base representations of n in which all the parts are multiples of 3.

Using the same notation as part (a), it suffices to show that there is a bijection between the two sets

$$S = \left\{(n_1, n_2, \ldots, n_r) \in \mathcal{P}_n : 3 \mid n_k \text{ for all } 1 \le k \le r\right\} \quad \text{and} \quad T = \mathcal{P}_{n/3}.$$

The key observations are as follows:

- If m is a $\{2,3\}$-integer that is a multiple of 3, then $\frac{m}{3}$ is a $\{2,3\}$-integer.
- If m is a $\{2,3\}$-integer, then $3m$ is a $\{2,3\}$-integer that is a multiple of 3.

Given a double-base representation $(n_1, \ldots, n_r)$ of n in which all parts are multiples of 3, note that $\left(\frac{n_1}{3}, \ldots, \frac{n_r}{3}\right)$ is a double-base representation of $\frac{n}{3}$. To see this, note that $n_1 < \cdots < n_r$ implies $\frac{n_1}{3} < \cdots < \frac{n_r}{3}$, and $\frac{n_x}{3}$ is a $\{2,3\}$-integer for all $1 \le x \le r$ using the key observations.

Similarly, given a double-base representation $(n_1, \ldots, n_r)$ of $\frac{n}{3}$, note that $(3n_1, \ldots, 3n_r)$ is a double-base representation of n in which all parts are multiples of 3. Again, to see this, note that $n_1 < \cdots < n_r$ implies $3n_1 < \cdots < 3n_r$, and $3n_x$ is a $\{2,3\}$-integer that is a multiple of 3 for all $1 \le x \le r$ using the key observations.

Pairing up the elements of S with elements of T as described above yields a bijection between S and T. Therefore $|S| = |T| = f\left(\frac{n}{3}\right)$, hence $f(n) = f(n-1) + f\left(\frac{n}{3}\right)$ if n is a multiple of 3, as desired.

8. The given statement is true if and only if k is a $\{2,3\}$-integer and $z = 0$ or z is a factor of k.

First suppose k is not a $\{2,3\}$-integer. Let $n > z$ and note that any double-base representation of n in which each part is either z or a multiple of k must contain at least one multiple of k. Because all parts of a strictly-chained double-base representation are $\{2,3\}$-integers (and hence any factor of a part is also a $\{2,3\}$-integer), it follows that there are no strictly-chained double-base representations of n in which each part is equal to either z or a multiple of k. However, $\frac{n-z}{k} > 0$ so $p\left(\frac{n-z}{k}\right) > 0$, and therefore the statement does not hold if k is not a $\{2,3\}$-integer.

Now suppose k is a $\{2,3\}$-integer and $z > 0$ is *not* a factor of k. Consider the strictly-chained double-base representations of $n = z + k$ in which each part is either z or a multiple of k. Note that such representations must have z as one of the parts because n is not a multiple of k. Furthermore, such representations must have at least one part that is a multiple of k because $n > z$. Therefore the only such double-base representation of n would be (z, k), but this is not a strictly-chained representation because $z \nmid k$. However, $p\left(\frac{n-z}{k}\right) = p(1) = 1$, and therefore the statement does not hold if k is a $\{2,3\}$-integer and $z > 0$ is not a factor of k.

Finally suppose k is a $\{2,3\}$-integer and $z = 0$ or $z \mid k$. Let $\mathcal{Q}_n$ be the set of strictly-chained double-base representations $(n_1, \ldots, n_r)$ of n. Letting n be a positive integer such that $n - z$ is a multiple of k, it suffices to show a bijection between the two sets

$$S = \left\{(n_1, n_2, \ldots, n_r) \in \mathcal{Q}_n : n_x = z \text{ or } k \mid n_x \text{ for all } 1 \le x \le r\right\} \quad \text{and} \quad T = \mathcal{Q}_{(n-z)/k}.$$

The key observations are as follows:

- If m is a $\{2,3\}$-integer that is a multiple of k, then $\frac{m}{k}$ is a $\{2,3\}$-integer.
- If m is a $\{2,3\}$-integer, then km is a $\{2,3\}$-integer that is a multiple of k.
- If $z = 0$, then all parts of a double-base representation in S would be multiples of k, as z *cannot* be one of the parts in a double-base representation of n (because 0 is not a $\{2,3\}$-integer).
- If $z > 0$, then any double-base representation $(n_1, \ldots, n_r) \in S$ satisfies $n_1 = z$ and n_x is a multiple of k for all $2 \le x \le r$. To see why, note that n is not a multiple of k if $z > 0$, and therefore not all the n_i's are multiples of k. It follows that $n_x = z$ for some $1 \le x \le r$, and in particular, $n_1 = z$ because $z < k$ (this implies n_x is a multiple of k for all $2 \le x \le r$).

First suppose $z = 0$. Given a strictly-chained double-base representation $(n_1, \ldots, n_r) \in S$ (so n_x is a multiple of k for all $1 \le x \le r$), note that $\left(\frac{n_1}{k}, \ldots, \frac{n_r}{k}\right)$ is a strictly-chained double-base representation of $\frac{n}{k} = \frac{n-z}{k}$. Indeed, using the key observations, it simply suffices to check that $\frac{n_x}{k}$ divides $\frac{n_{x+1}}{k}$ for all $1 \le x < r$; this follows from the fact that n_x divides n_{x+1} for all $1 \le x < r$. Similarly, given a strictly-chained double-base representation $(n_1, \ldots, n_r)$ of $\frac{n}{k} = \frac{n-z}{k}$, note that $(kn_1, \ldots, kn_r)$ is a strictly-chained double-base representation of n in which all parts are equal to either z or a multiple or k.

Now suppose $z > 0$. Note that if $n = z$, there is exactly one element of S, namely (z). Furthermore, $p\left(\frac{n-z}{k}\right) = p(0) = 1$, so it suffices to show the result also holds for $n > z$. Given a strictly-chained double-base representation $(n_1, \ldots, n_r) = (z, n_2, \ldots, n_r) \in S$ (so $n_1 = z$ and n_x is a multiple of k for all $2 \le x \le r$), note that $\left(\frac{n_2}{k}, \ldots, \frac{n_r}{k}\right)$ is a strictly-chained double-base representation of $\frac{n-z}{k}$. Similarly, given a strictly-chained double-base representation $(n_1, \ldots, n_r)$ of $\frac{n-z}{k}$, note that $(z, kn_1, \ldots, kn_r)$ is a strictly-chained double-base representation of n (because $z \mid k$ so $z \mid kn_1$) in which all parts are equal to either z or a multiple or k.

Pairing up the elements of S and T as described above in either the $z = 0$ or $z > 0$ cases yields a bijection between S and T, so $|S| = |T| = p\left(\frac{n-z}{k}\right)$. Hence if n is a positive integer such that $n - z$ is a multiple of k (with $0 \le z < k$), there are exactly $p\left(\frac{n-z}{k}\right)$ strictly-chained double-base representations of n in which all parts are multiples of k if and only if k is $\{2, 3\}$-integer and either $z = 0$ or z is a factor of k.

9. (a) Any strictly-chained double-base representation of $6n+1$ contains a 1 because $6n+1$ is neither a multiple of 2 nor 3. Let $(m_1, \ldots, m_r)$ be a strictly-chained double-base representation of $6n + 1$ in which $m_1 = 1$.

Let A, B, C denote the numbers of strictly-chained double-base representations of $6n+1$ in which $m_1 = 1$ and m_2 is a multiple of 2, 3, or 6, respectively. The number of strictly-chained double-base representations of $6n+1$ is equal to the number of strictly-chained double-base representations of $6n+1$ in which $m_1 = 1$ and m_2 is either a multiple of 2 or a multiple of 3. By the Principle of Inclusion-Exclusion, there are $A + B - C$ strictly-chained double-base representations of $6n + 1$, thus $p(6n + 1) = A + B - C$.

For $k \in \{2, 3, 6\}$, there are exactly $p\left(\frac{6n}{k}\right)$ such strictly-chained double-base representations in which m_2 is a multiple of k. To see this, if m_2 is a multiple of k, then m_x is a multiple of k for all $2 \le x \le r$. Therefore the number of strictly-chained double-base representations of $6n+1$ in which $m_1 = 1$ and m_2 is a multiple of k is equal to the number of strictly-chained double-base representations of $6n$ in which all parts are multiples of k. By Problem 8, there are exactly $p\left(\frac{6n}{k}\right)$ strictly-chained double-base representations of $6n + 1$ in which all parts are either 1 or multiples of k.

Hence $p(6n + 1) = p(3n) + p(2n) - p(n)$, as desired.

(b) By Problem 8, there are exactly $p\left(\frac{n}{3}\right)$ strictly-chained double-base representations of n in which all parts are multiples of 3. It therefore suffices to show that there are exactly $p(n-1)$ strictly-chained double-base representations of n in which not all parts are multiples of 3. The proof is very similar to the proof of Problem 7(a). Using the same notation from Problem 8, it suffices to show a bijection between the two sets

$$S = \left\{(n_1, n_2, \ldots, n_r) \in \mathcal{Q}_n : 3 \nmid n_1\right\} \quad \text{and} \quad T = \mathcal{Q}_{n-1}.$$

Consider some $(n_1, \ldots, n_r) \in S$. In other words, $(n_1, \ldots, n_r)$ is a double-base representation of n such that n_1 is not a multiple of 3 and n_x divides n_{x+1} for all $1 \le x < r$. By the divisibility condition, one may assume $n_1, \ldots, n_j$ are not multiples of 3 while $n_{j+1}, \ldots, n_r$ are multiples of 3 for some $1 \le j \le r$. Then $n_1 + \cdots + n_j$ is a binary representation of $n - (n_{j+1} + \cdots + n_r) \ge 3$ (the inequality holds because the left side is a positive multiple of 3). Let $x_1 + x_2 + \cdots + x_m$ be the binary representation of $n - (n_{j+1} + \cdots + n_r) - 1$ such that $x_1 < x_2 < \cdots < x_m$.

Then consider the double-base representation $(x_1, \ldots, x_m, n_{j+1}, \ldots, n_r)$ of $n - 1$. Note that x_m divides n_{j+1}. To see this, note that x_m is the greatest power of 2 in the binary representation of $n - (n_{j+1} + \cdots + n_r) - 1$ while n_j is the greatest power of 2 in the binary representation of $n - (n_{j+1} + \cdots + n_r)$. Therefore $x_m \le n_j$ and in particular, x_m divides n_j because they are both powers of 2. It follows that the above double-base representation is also a strictly-chained double-base representation of $n - 1$, and therefore $(x_1, \ldots, x_m, n_{j+1}, \ldots, n_r) \in T$.

Furthermore, this mapping is invertible. Consider $(n_1, \ldots, n_r) \in T$ where for some $1 \le k \le r$, the following holds: n_i is a power of 2 for all $1 \le i \le k$ and n_i is a multiple of 3 for all $k < i \le r$. Note that n_t is a factor of n_{t+1} for all $1 \le t < r$.

In particular, $n_1 + \cdots + n_k$ is the binary representation of $n - 1 - (n_{k+1} + \cdots + n_r)$. Let $y_1 + \cdots + y_{m'}$ be the binary representation of $n - (n_{k+1} + \cdots + n_r)$ such that $y_1 < y_2 < \cdots < y_{m'}$. Then $(y_1, \ldots, y_{m'}, n_{k+1}, \ldots, n_r)$ is a double-base representation of n in which $3 \nmid y_1$ (because $y_1 + \cdots + y_{m'}$ is a nonempty binary representation). Note that $y_{m'}$ divides n_{k+1}. To see this, note that $y_{m'}$ is the greatest power of 2 in the binary representation of $n - (n_{k+1} + \cdots + n_r)$ while n_k is the greatest power of 2 in the binary representation of $n - (n_{k+1} + \cdots + n_r) - 1$. Therefore $y_{m'} \geq n_k$. If $y_{m'} > n_k$, then $n - (n_{k+1} + \cdots + n_r)$ is a power of 2, which is a contradiction because it is a multiple of 3. Hence $y_{m'} = n_k$. so $y_{m'}$ divides n_{k+1}. It follows that the above double-base representation is also a strictly-chained double-base representation of n, and therefore $(y_1, \ldots, y_{m'}, n_{k+1}, \ldots, n_r) \in S$. Note that this mapping is the inverse of the previous mapping.

Hence there are exactly $p(n-1)$ strictly-chained double-base representations of n in which not all parts are multiples of 3, and therefore $p(n) = p(n-1) + p\left(\frac{n}{3}\right)$ if n is a multiple of 3, as desired.

(c) Let $(n_1, n_2, \ldots, n_r)$ be a strictly-chained double-base representation of n.

First note that none of the n_i's can be a power of 3 greater than 1. To see this, if one of the n_i's were such a power of 3, then either $n_1 = 3^k$ or $(n_1, n_2) = (1, 3^k)$, for some positive integer k, due to the divisibility constraint. However, this implies that $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$, which is a contradiction.

Thus every part of the strictly-chained double-base representation of n (except for possibly n_1, which could equal 1) is even. Now consider the parity of n.

If n is odd, then $n_1 = 1$ and every other part is even for any strictly-chained double-base representation of n. Therefore the number of strictly-chained double-base representations of n equals the number of strictly-chained double-base representations of n in which all parts are either equal to 1 or are even. By Problem 8, there are $p\left(\frac{n-1}{2}\right) = p\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$ such strictly-chained double-base representations.

If n is even, then every part is even for any strictly-chained double-base representation of n. Therefore the number of strictly-chained double-base representations of n is equal to the number of strictly-chained double-base representations of n in which all parts are even. By Problem 8, there are $p\left(\frac{n}{2}\right) = p\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$ such strictly-chained double-base representations.

Hence if n leaves a remainder of 2 when divided by 3, then $p(n) = p\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$, as desired.

(d) First suppose that n is odd; then $n = 2k + 1$ for some nonnegative integer k. Then

$$3 \cdot \left\lfloor \frac{n+1}{2} \right\rfloor - 1 = 3 \cdot \left\lfloor \frac{2k+2}{2} \right\rfloor - 1 = 3(k+1) - 1 = 3k + 2,$$

so it suffices to show $p(6k+4) = p(2k+1) + p(3k+2)$.

There are exactly $p(2k+1)$ strictly-chained double-base representations of $6k+4$ that contain a 1. To see this, let $(m_1, \ldots, m_r)$ be a strictly-chained double-base representation of $6k+4$ in which $m_1 = 1$. If m_2 is even, then m_x is even for all $2 \leq x \leq r$, and therefore $6k+4$ is odd, which is a contradiction. Thus m_2 is a multiple of 3, so m_x is a multiple of 3 for all $1 \leq x \leq r$. In other words, the number of strictly-chained double-base representations of $6k+4$ that contain a 1 is equal to the number of strictly-chained double-base representations of $6k+4$ in which all parts are either equal to 1 or are multiples of 3. By Problem 8, there are $p(2k+1)$ strictly-chained double-base representations of $6k+4$ that contain a 1.

Furthermore, there are exactly $p(3k+2)$ strictly-chained double-base representations of $6k+4$ that do not contain a 1. To see this, let $(m_1, \ldots, m_r)$ be a strictly-chained double-base representation of $6k+4$ that does not contain a 1. If m_1 is a multiple of 3, then m_x is a multiple of 3 for all $1 \leq x \leq r$, and therefore $6k+4$ is a multiple of 3, which is a contradiction. Also, $m_1 \neq 1$, so m_1 is even and therefore m_x is even for all $1 \leq x \leq r$. By Problem 8, there are exactly $p\left(\frac{6k+4}{2}\right) = p(3k+2)$ strictly-chained double-base representations of $6k+4$ in which all the parts are even.

Now suppose that n is even; then $n = 2k$ for some positive integer k. Then

$$3 \cdot \left\lfloor \frac{n+1}{2} \right\rfloor - 1 = 3 \cdot \left\lfloor \frac{2k+1}{2} \right\rfloor - 1 = 3k - 1,$$

so it suffices to show that $p(6k+1) = p(2k) + p(3k-1)$. From part (a) and part (b), it follows that

$$\begin{aligned} p(6k+1) &= p(3k) + p(2k) - p(k) \\ &= \big(p(k) + p(3k-1)\big) + \big(p(2k) - p(k)\big) \\ &= p(2k) + p(3k-1), \end{aligned}$$

hence $p(3n+1) = p(n) + p\left(3 \cdot \left\lfloor \frac{n+1}{2} \right\rfloor - 1\right)$ for all positive integers n, as desired.

10. There are **61** strictly-chained double-base representations of 2025. To see this, one may use the recurrence relations in Problem 9, as shown below. Using Problem 9(b) repeatedly, it follows that

$$\begin{aligned} p(2025) &= p(2024) + p(675) \\ &= p(2024) + p(674) + p(225) \\ &= p(2024) + p(674) + p(224) + p(75) \\ &= p(2024) + p(674) + p(224) + p(74) + p(25). \end{aligned}$$

and using the result from Problem 9(c) repeatedly, it follows that $p(2025) = p(1012) + p(337) + p(112) + p(37) + p(25)$. Note that using the result from Problem 9(d) repeatedly yields

$$\begin{aligned} p(1012) &= p(3 \cdot 337 + 1) = p(337) + p\left(3 \cdot \left\lfloor \frac{338}{2} \right\rfloor - 1\right) = p(337) + p(506) \\ p(337) &= p(3 \cdot 112 + 1) = p(112) + p\left(3 \cdot \left\lfloor \frac{113}{2} \right\rfloor - 1\right) = p(112) + p(167) \\ p(112) &= p(3 \cdot 37 + 1) = p(37) + p\left(3 \cdot \left\lfloor \frac{38}{2} \right\rfloor - 1\right) = p(37) + p(56) \\ p(37) &= p(3 \cdot 12 + 1) = p(12) + p\left(3 \cdot \left\lfloor \frac{13}{2} \right\rfloor - 1\right) = p(12) + p(17) \\ p(25) &= p(3 \cdot 8 + 1) = p(8) + p\left(3 \cdot \left\lfloor \frac{9}{2} \right\rfloor - 1\right) = p(8) + p(11). \end{aligned}$$

The result from Problem 9(c) gives $p(1012) = p(337) + p(253)$ and $p(337) - p(112) = p(83) = p(41) = p(20) = p(10)$. Similarly, $p(112) = p(37) + p(28)$ and $p(37) = p(12) + p(8)$. Note that $p(12) = p(4) + p(11) = p(4) + p(5)$ and using the result from Problem 9(d) repeatedly yields

$$\begin{aligned} p(253) &= p(3 \cdot 84 + 1) = p(84) + p\left(3 \cdot \left\lfloor \frac{85}{2} \right\rfloor - 1\right) = p(84) + p(125) \\ p(28) &= p(3 \cdot 9 + 1) = p(9) + p\left(3 \cdot \left\lfloor \frac{10}{2} \right\rfloor - 1\right) = p(9) + p(14) \end{aligned}$$

Using the result from Problem 9(c), $p(125) = p(62) = p(31)$ and Problem 9(d) implies $p(31) = p(10) + p(14) = p(10) + p(7)$. Furthermore, using $p(83) = p(10)$ and $p(28) = p(9) + p(14) = p(9) + p(7)$ from above, the result from Problem 9(b) implies $p(84) = p(28) + p(83) = p(9) + p(7) + p(10)$.

To summarize the above computations, the following equalities hold:

$$\begin{aligned} p(2025) &= p(1012) + p(337) + p(112) + p(37) + p(25) \\ p(1012) &= p(337) + [p(9) + p(7) + p(10)] + [p(10) + p(7)] \\ p(337) &= p(112) + p(10) \\ p(112) &= p(37) + [p(9) + p(7)] \\ p(37) &= p(4) + p(5) + p(8) \\ p(25) &= p(8) + p(5). \end{aligned}$$

One can create a table of values for $p(n)$ when $n \leq 10$:

n	$p(n)$
1	1
2	1
3	2
4	2
5	1

n	$p(n)$
6	2
7	2
8	2
9	4
10	3

Therefore $p(25) = 2 + 1 = 3$ and $p(37) = 2 + 1 + 2 = 5$. It follows that $p(112) = 5 + (4 + 2) = 11$ and $p(337) = 11 + 3 = 14$. This gives $p(1012) = 14 + (4 + 2 + 3) + (3 + 2) = 28$, and these values yield $p(2025) = 3 + 5 + 11 + 14 + 28 = 61$.

Below are some interesting facts about extensions of this Power Question for the interested reader.

- There exists a constant $C > 0$ such that for sufficiently large n, there is a double-base representation of n with length at most $\dfrac{C \log n}{\log(\log n)}$. A proof of this result can be found in the paper "Theory and Applications for a Double-Base Number System" by Dimitrov et al.
- There exists a constant $C > 0$ such that for infinitely many integers n, every signed double-base representation of n has length greater than $\dfrac{C \log n}{(\log(\log n)) \cdot (\log(\log(\log n)))}$. A proof of this result can be found in the paper "Lower Bounds on the Lengths of Double-Base Representations" by Dimitrov et al.
- There exists a constant $C > 0$ such that for all sufficiently large n, there is a $\{2, 3\}$-integer between $n - \dfrac{n}{(\log n)^C}$ and n. A proof of this result can be found in the paper "On the maximal distance between integers composed of small primes" by Tijdeman.
- The number of double-base representations of n, $f(n)$, is asymptotically equal to

$$\Theta\left((\log n)^{C_1} n^{C_2} \exp\left(C_3 \log^2\left(\frac{n}{\log n}\right)\right)\right)$$

  for some constants C_1, C_2, C_3. Note that $g(n) = \Theta(h(n))$ means that there are constants X and Y for which $X \cdot h(n) < g(n) < Y \cdot h(n)$ for sufficiently large n. A proof of this result (as well as generalizations) can be found in the paper "Multi-Base Representations of Integers: Asymptotic Enumeration and Central Limit Theorems" by Krenn et al.
- Double-base representations have applications in cryptography (in particular, Elliptic Curve Cryptography). For example, see the paper "Extended Double-Base Number System with applications to Elliptic Curve Cryptography" by Doche et al.

6 Individual Round

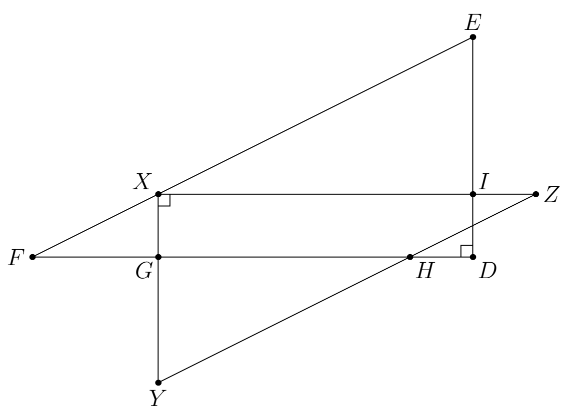
Problem 1. Compute the least integer N for which $\dfrac{N!}{(N+3)(N+4)(N+5)}$ is an integer.

Problem 2. Compute the number of ways in which an 87×1 grid can be fully tiled with rectangles of size 8×1 and 7×1.

Problem 3. Compute the number of positive integers N for which $0.\overline{20} + \dfrac{1}{N} > 0.\overline{25}$.

Problem 4. The number $\sqrt{5} \cdot \dfrac{\sqrt{3+\sqrt{5}} + \sqrt{3-\sqrt{5}}}{\sqrt{3-\sqrt{8}}}$ can be expressed as $a + b\sqrt{c}$, where a, b, and c are positive integers and c is squarefree. Compute the ordered triple (a, b, c).

Problem 5. Similar right triangles DEF and XYZ have right angles at D and X, respectively. Let $\overline{DF}$ intersect $\overline{XY}$ at G, let $\overline{DF}$ intersect $\overline{YZ}$ at H, and let $\overline{DE}$ intersect $\overline{XZ}$ at I. Segments $\overline{EF}$ and $\overline{YZ}$ are parallel, and X lies on $\overline{EF}$ so that $XG = ZI = DH$. Given that $YG = FG = 8$, compute the greater of $[DEF]$ and $[XYZ]$.



Problem 6. Compute the least positive multiple of 7 (expressed as a base-ten number) that has exactly seven 1s in its binary representation.

Problem 7. Let x satisfy $\log_{50^{(50^2)}}\left(\log_{2^{(50^{50})}} 2^x\right) = \frac{1}{100}$. Compute $\log_{50} x$.

Problem 8. The polynomial f has integer coefficients, and

$$f\left(\sqrt{20} + \sqrt{2} + 5\right) = 2025.$$

Compute the least possible *positive* value of $f(1)$.

Problem 9. Compute the number of permutations of the digits of 123456789 for which the odd digits are in increasing order.

Problem 10. Let $\triangle ABC$ be an acute triangle with incenter I. Suppose $\csc A$, $\csc B$, and $\csc C$ form an arithmetic progression with common difference $\frac{1}{20}$, and AI^2, BI^2, and CI^2 form an arithmetic progression with common difference 25. Compute $[ABC]$.

7 Individual Round Answers

Answer 1. 11

Answer 2. 231

Answer 3. 19

Answer 4. $(10, 5, 2)$

Answer 5. 196

Answer 6. 623

Answer 7. 75

Answer 8. 41

Answer 9. 3024

Answer 10. 250

8 Individual Round Solutions

Problem 1. Compute the least integer N for which $\dfrac{N!}{(N+3)(N+4)(N+5)}$ is an integer.

Solution 1. Note that if any of $N+3$, $N+4$, and $N+5$ is prime, then the given fraction will not be an integer because that prime will not divide $N!$. Therefore look for three consecutive composite numbers. The least three consecutive composite numbers are 8, 9, and 10, but if $N+5=10$, then $N=5$, and $5! = 120 < 720 = 8\cdot 9\cdot 10$. The next least three consecutive composite numbers are 14, 15, 16, and because $\frac{11!}{14\cdot 15\cdot 16} = 11880$ is an integer, the answer is **11**.

Problem 2. Compute the number of ways in which an 87×1 grid can be fully tiled with rectangles of size 8×1 and 7×1.

Solution 2. The tiling must use at least $\left\lceil \frac{87}{8}\right\rceil = 11$ tiles, and at most $\left\lfloor \frac{87}{7}\right\rfloor = 12$ tiles. If there are 11 tiles, then there must be ten 8×1 tiles and one 7×1 tile; there are $\binom{11}{1} = 11$ such tilings, because the order of the tiles matters. If there are 12 tiles, then there must be three 8×1 tiles and nine 7×1 tiles. In this case, there are $\binom{12}{3} = 220$ such tilings. Thus the total number of tilings is $11 + 220 = \mathbf{231}$.

Problem 3. Compute the number of positive integers N for which $0.\overline{20} + \dfrac{1}{N} > 0.\overline{25}$.

Solution 3. Notice that $0.\overline{25} = \frac{25}{99}$ and $0.\overline{20} = \frac{20}{99}$, so it follows that $\frac{1}{N} > \frac{5}{99} \rightarrow N < \frac{99}{5}$. Thus $N < 19.8$, so the answer is **19**.

Problem 4. The number $\sqrt{5}\cdot\dfrac{\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}}{\sqrt{3-\sqrt{8}}}$ can be expressed as $a + b\sqrt{c}$, where a, b, and c are positive integers and c is squarefree. Compute the ordered triple (a, b, c).

Solution 4. Because

$$\left(\sqrt{2}-1\right)^2 = 2+1-2\sqrt{2} = 3-2\sqrt{2} = 3-\sqrt{8},$$

it follows that

$$\sqrt{3-\sqrt{8}} = \sqrt{2}-1 > 0.$$

Let $S = \sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}$. It follows that

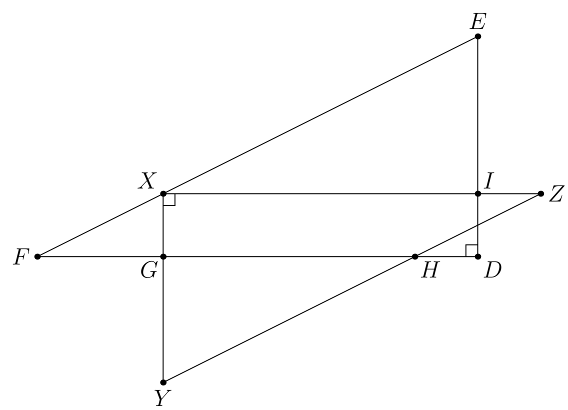
$$\begin{aligned} S^2 &= \left(\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}\right)^2 \\ &= \left(3+\sqrt{5}\right)+\left(3-\sqrt{5}\right)+2\cdot\left(\sqrt{3+\sqrt{5}}\right)\cdot\left(\sqrt{3-\sqrt{5}}\right) \\ &= 6+2\sqrt{9-5} = 10, \end{aligned}$$

so $S = \sqrt{10}$. Finally, the desired fraction can be written as

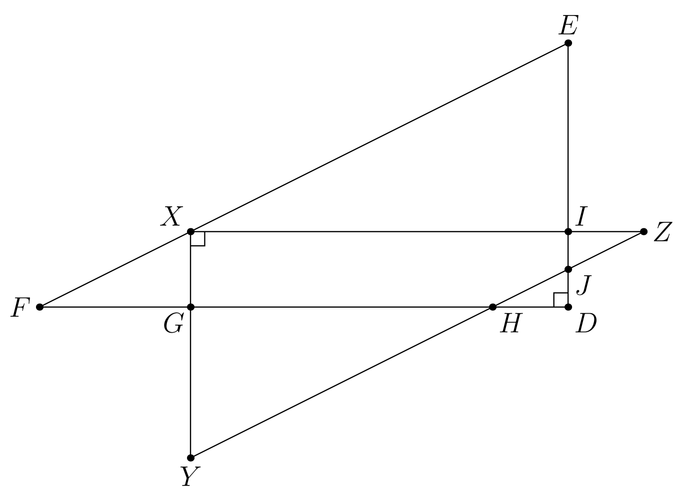
$$\begin{aligned} \sqrt{5}\cdot\frac{\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}}{\sqrt{3-\sqrt{8}}} &= \sqrt{5}\cdot\frac{\sqrt{10}}{\sqrt{2}-1} \\ &= \frac{\sqrt{50}(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} \\ &= \sqrt{100}+\sqrt{50} = 10+5\sqrt{2}. \end{aligned}$$

Thus the desired ordered triple is $(a, b, c) = \mathbf{(10, 5, 2)}$.

Problem 5. Similar right triangles DEF and XYZ have right angles at D and X, respectively. Let $\overline{DF}$ intersect $\overline{XY}$ at G, let $\overline{DF}$ intersect $\overline{YZ}$ at H, and let $\overline{DE}$ intersect $\overline{XZ}$ at I. Segments $\overline{EF}$ and $\overline{YZ}$ are parallel, and X lies on $\overline{EF}$ so that $XG = ZI = DH$. Given that $YG = FG = 8$, compute the greater of $[DEF]$ and $[XYZ]$.



Solution 5. Let $\overline{YZ}$ intersect $\overline{DE}$ at J, as shown below.



Let $XG = ZI = DH = x$. By parallel lines, it follows that $\triangle XGF \sim \triangle YGH \sim \triangle JDH \sim \triangle JIZ$. Furthermore, because $ZI = DH$, it follows that $\triangle JDH \cong \triangle JIZ$. Thus $JD = JI$. Because $XG = ID$, it follows that $JD = JI = x/2$. Then $XG/FG = JD/HD = 1/2$, so $x = 8 \cdot \frac{1}{2} = 4$.

Now $YG/HG = 1/2$, so $HG = 16$, and so $DF = 28$. Also $XI = GD = 20$, so $XZ = 24$. So the larger of $\triangle DEF$ and $\triangle XYZ$ is $\triangle DEF$. To compute $[DEF]$, it suffices to find DE because DF is already known.

Because $EI/XI = 1/2$, it follows that $EI = 10$, and so $DE = 14$. Thus $[DEF] = \frac{1}{2} \cdot 28 \cdot 14 = \mathbf{196}$.

Problem 6. Compute the least positive multiple of 7 (expressed as a base-ten number) that has exactly seven 1s in its binary representation.

Solution 6. If a positive integer n has exactly seven 1s in its binary representation, it must be equal to $2^{a_1} + 2^{a_2} + \cdots + 2^{a_7}$ for distinct nonnegative integers $a_1 > a_2 > \cdots > a_7$. The least possible value of a_1 is 6. Check the possible values of a_1 in increasing order.

Suppose that $a_1 = 6$. Then $(a_1, a_2, \ldots, a_7) = (6, 5, 4, 3, 2, 1, 0)$. So $n = 2^6 + 2^5 + \cdots + 2^0 = 127$. This is not a multiple of 7.

Suppose that $a_1 = 7$. Then $n = 2^7 + 2^6 + \cdots + 2^0 - 2^b = 255 - 2^b$ for some $b \in \{0, \ldots, 6\}$. Then $n \equiv 3 - 2^b \pmod 7$. But $2^b \pmod 7 \in \{1, 2, 4\}$, so n is not a multiple of 7.

Suppose that $a_1 = 8$. Then $n = 2^8 + 2^7 + \cdots + 2^0 - 2^b - 2^c = 511 - 2^b - 2^c$ for some distinct $b, c \in \{0, \ldots, 7\}$. Then $n \equiv -2^b - 2^c \pmod 7$. But $2^b \pmod 7 \in \{1, 2, 4\}$ and $2^c \pmod 7 \in \{1, 2, 4\}$, so n is not a multiple of 7.

Suppose that $a_1 = 9$. Then $n = 2^9 + 2^8 + \cdots + 2^0 - 2^b - 2^c - 2^d = 1023 - 2^b - 2^c - 2^d$ for some distinct b, c, $d \in \{0, \ldots, 8\}$. Then $n \equiv 1 - 2^b - 2^c - 2^d \pmod 7$. In order for n to be a multiple of 7, this requires 2^b, 2^c, and $2^d \pmod 7$ to be equivalent to 2, 2, and 4 (mod 7) in some order. To find the least possible n, it suffices to make $2^b + 2^c + 2^d$ as great as possible. The values of $k \in \{0, \ldots, 8\}$ such that $2^k \equiv 2 \pmod 7$ are 1, 4, and 7. The values of $k \in \{0, \ldots, 8\}$ such that $2^k \equiv 4 \pmod 7$ are 2, 5, and 8. Thus the greatest possible value of $2^b + 2^c + 2^d$ under these conditions is $2^8 + 2^7 + 2^4$. It follows that the least possible n is $1023 - 2^8 - 2^7 - 2^4 = \mathbf{623}$.

Problem 7. Let x satisfy $\log_{50^{(50^2)}}\left(\log_{2^{(50^{50})}} 2^x\right) = \frac{1}{100}$. Compute $\log_{50} x$.

Solution 7. Let $\log_{50} x = y$. Then $x = 50^y$, so $\log_{2^{(50^{50})}} 2^x = \frac{x}{50^{50}} = \frac{50^y}{50^{50}} = 50^{y-50}$. Thus $\log_{50^{(50^2)}}\left(\log_{2^{(50^{50})}} 2^x\right) = \log_{50^{(50^2)}} 50^{y-50} = \frac{y-50}{2500}$. Setting this equal to $\frac{1}{100}$ yields the equation $y - 50 = 25$, so $y = \mathbf{75}$.

Problem 8. The polynomial f has integer coefficients, and

$$f\left(\sqrt{20} + \sqrt{2} + 5\right) = 2025.$$

Compute the least possible *positive* value of $f(1)$.

Solution 8. Take the minimal polynomial[1] of $\sqrt{20} + \sqrt{2} + 5$, namely

$$g(X) = \left((X-5)^2 - 22\right)^2 - 160.$$

(Note that the four roots of this polynomial are $\pm\sqrt{20} \pm \sqrt{2} + 5$.) It must divide $f(X) - 2025$, meaning that

$$f(X) = 2025 + g(X) \cdot q(X),$$

for some polynomial q with integer coefficients. Thus $f(1) = 2025 + g(1) \cdot q(1)$. However,

$$g(1) = \left((1-5)^2 - 22\right)^2 - 160 = (16 - 22)^2 - 160 = -124.$$

Hence the necessary and sufficient condition for possible values of $f(1)$ is

$$f(1) \equiv 2025 \equiv 41 \pmod{124}.$$

The answer is thus **41**.

Problem 9. Compute the number of permutations of the digits of 123456789 for which the odd digits are in increasing order.

Solution 9. The positions of the odd digits determine their individual locations, and there are $\binom{9}{5} = \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1} = 126$ ways to place them. The even digits can then be arranged in $4! = 24$ ways for a total of $126 \cdot 24 = \mathbf{3024}$ permutations whose odd digits are in increasing order.

[1] The *minimal polynomial* of a complex number z is the least-degree monic polynomial with rational coefficients that has z as a root (if such a polynomial exists).

Problem 10. Let $\triangle ABC$ be an acute triangle with incenter I. Suppose $\csc A$, $\csc B$, and $\csc C$ form an arithmetic progression with common difference $\frac{1}{20}$, and AI^2, BI^2, and CI^2 form an arithmetic progression with common difference 25. Compute $[ABC]$.

Solution 10. Let $a = BC$, $b = CA$, $c = AB$, $s = (a+b+c)/2$, and let r be the inradius of $\triangle ABC$. Note that

$$\begin{aligned} AI^2 - BI^2 &= \left((s-a)^2 + r^2\right) - \left((s-b)^2 + r^2\right) \\ &= \left((s-a) + (s-b)\right) \cdot \left((s-a) - (s-b)\right) \\ &= c \cdot (b-a) \\ &= cb - ca \\ &= 2[ABC] \cdot \left(\frac{1}{\sin A} - \frac{1}{\sin B}\right) \\ &= 2[ABC] \cdot (\csc A - \csc B)\,. \end{aligned}$$

Hence

$$[ABC] = \frac{AI^2 - BI^2}{2(\csc A - \csc B)} = \frac{25}{2 \cdot \frac{1}{20}} = \mathbf{250}.$$

Remark: This solution does not use the entire arithmetic progression. In fact, it shows that as long as $\csc A$, $\csc B$, $\csc C$ are an arithmetic progression, then the lengths AI^2, BI^2, CI^2 are an arithmetic progression as well, with the common difference scaled by $2[ABC]$. Consequently, even though the problem seems to specify four conditions, one of them is redundant. The unique choice of angles that yield the desired progression are $\angle A \approx 64.457°$, $\angle B \approx 59.691°$, $\angle C \approx 55.852°$.

9 Relay Round

Problem 1-1. Compute the greatest integer N such that 50^N is a divisor of $(20!)(25!)$.

Problem 1-2. *Let T be the number you will receive.* Square $ABCD$ has area 50. Points P and Q lie on $\overline{AB}$ and $\overline{BC}$, respectively, so that $BP = BQ = \sqrt{2T}$. Points R and S lie inside square $ABCD$ so that $PQRS$ is a square. Let M be the midpoint of $\overline{RS}$. The length DM can be expressed as $x - y\sqrt{z}$, where x, y, and z are integers and z is squarefree. Compute xyz.

Problem 1-3. *Let T be the number you will receive.* In $\triangle ABC$, $\angle C$ is a right angle. Point M is the midpoint of $\overline{BC}$, and point N is the midpoint of $\overline{AC}$. Given that $AM^2 + BN^2 = T$, compute AB.

Problem 2-1. Regular octagon $SHIPYARD$ and regular heptagon $CENTRAL$ share side $\overline{RA}$, and point E lies inside $SHIPYARD$. Compute the integer closest to the degree measure of $\angle DRT$.

Problem 2-2. *Let T be the number you will receive.* Let $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$. Compute the greatest prime factor of C_T.

Problem 2-3. *Let T be the number you will receive.* A sequence is defined by $C_0 = C_1 = 1$, and $C_n = C_0C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \cdots + C_{n-1}C_0$ for $n > 1$. Compute the number of nonnegative integers $k < 2^T$ such that C_k is odd.

10 Relay Round Answers

Answer 1-1. 5

Answer 1-2. 150

Answer 1-3. $2\sqrt{30}$

Answer 2-1. 6

Answer 2-2. 11

Answer 2-3. 12

11 Relay Round Solutions

Problem 1-1. Compute the greatest integer N such that 50^N is a divisor of $(20!)(25!)$.

Solution 1-1. Let $P = (20!)(25!)$. Each of the factors 5, 10, 15, 20 in 20! and 25! contributes a factor of 5 to P, and the factor of 25 in 25! contributes two factors of 5 to P. Thus P contains $2 \cdot 4 + 2 = 10$ factors of 5, and because 50 has two factors of 5, the answer is $N = 10/2 = \mathbf{5}$. Note that 50^5 has five factors of 2, and 32 is a divisor of P.

Problem 1-2. *Let T be the number you will receive.* Square $ABCD$ has area 50. Points P and Q lie on $\overline{AB}$ and $\overline{BC}$, respectively, so that $BP = BQ = \sqrt{2T}$. Points R and S lie inside square $ABCD$ so that $PQRS$ is a square. Let M be the midpoint of $\overline{RS}$. The length DM can be expressed as $x - y\sqrt{z}$, where x, y, and z are integers and z is squarefree. Compute xyz.

Solution 1-2. The side length of $ABCD$ is $\sqrt{50} = 5\sqrt{2}$, hence $BD = \sqrt{2} \cdot 5\sqrt{2} = 10$. Let O be the midpoint of $\overline{PQ}$. Note that $\triangle PBQ$ is a 45°–45°–90° triangle, so $m\angle PBO = m\angle QBO = 45^\circ$, and it follows that points O and M lie on diagonal $\overline{BD}$. Moreoever, note that $PQ = 2\sqrt{T}$ and $BO = \sqrt{T}$. Thus

$$DM = BD - BM = BD - (BO + OM) = 10 - \left(\sqrt{T} + 2\sqrt{T}\right) = 10 - 3\sqrt{T}.$$

With $T = 5$, it follows that $x = 10$, $y = 3$, and $z = 5$, so $xyz = \mathbf{150}$.

Problem 1-3. *Let T be the number you will receive.* In $\triangle ABC$, $\angle C$ is a right angle. Point M is the midpoint of $\overline{BC}$, and point N is the midpoint of $\overline{AC}$. Given that $AM^2 + BN^2 = T$, compute AB.

Solution 1-3. Let $BC = 2a$ and $AC = 2b$. Then $AM^2 + BN^2 = (a^2 + 4b^2) + (b^2 + 4a^2) = 5(a^2 + b^2) = T$. Thus $a^2 + b^2 = \frac{T}{5}$. Note that $AB^2 = 4a^2 + 4b^2 = \frac{4T}{5}$. With $T = 150$, $AB^2 = 120$, so $AB = \mathbf{2\sqrt{30}}$.

Problem 2-1. Regular octagon $SHIPYARD$ and regular heptagon $CENTRAL$ share side $\overline{RA}$, and point E lies inside $SHIPYARD$. Compute the integer closest to the degree measure of $\angle DRT$.

Solution 2-1. Note that $m\angle DRT = 180^\circ \cdot \left(\frac{6}{8} - \frac{5}{7}\right) = 180^\circ \cdot \frac{1}{28} \approx 6.4^\circ$, so the answer is **6**.

Problem 2-2. *Let T be the number you will receive.* Let $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$. Compute the greatest prime factor of C_T.

Solution 2-2. Note that $C_n = \frac{(2n)(2n-1)\cdots(n+3)(n+2)}{n(n-1)\cdots(2)(1)}$, and there is always a prime between n and $2n$, so the desired greatest prime factor is the greatest prime less than $2n = 2T$. With $T = 6$, the answer is **11**.

Problem 2-3. *Let T be the number you will receive.* A sequence is defined by $C_0 = C_1 = 1$, and $C_n = C_0C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \cdots + C_{n-1}C_0$ for $n > 1$. Compute the number of nonnegative integers $k < 2^T$ such that C_k is odd.

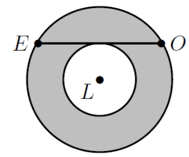
Solution 2-3. From the given recurrence, C_n is even when $n > 0$ is even because the first and second halves of the sum are identical. When n is odd, all terms appear twice except the middle term, $C^2_{(n-1)/2}$. It can be observed and then proved by induction that this happens if and only if n is 1 less than a power of 2. Thus C_n is nonnegative and odd for $n = 0, 2^1 - 1, 2^2 - 1, 2^3 - 1, \ldots, 2^T - 1$, which is a total of $T + 1$ nonnegative integers $k < 2^T$. With $T = 11$, the answer is **12**.

12 Super Relay

1. Compute the number of lattice points (points with integer coordinates) that lie on the graph of $x^2 + y^2 = 50$.

2. Let $T = $ TNYWR. For all nonnegative real numbers N and some value of K, the equation $\sqrt[K]{N^T} = \sqrt[50]{N^{24}}$ is an identity. Compute the numerical value of K.

3. Let $T = $ TNYWR. As shown, two circles are centered at point L. Points E and O lie on the larger circle so that chord $\overline{EO}$ is tangent to the smaller circle. The positive difference in the areas of the circles (shaded in the diagram) is $T\pi$. Compute EO.



4. Let $T = $ TNYWR. To celebrate the 50^{th} ARML, Renee baked $5T$ chocolate cookies and Richard baked $10T$ snowball cookies. Then Renee ate one of her cookies and Richard ate two of his cookies. One of the remaining cookies was then randomly selected. Compute the probability that it was a chocolate cookie.

5. Let $T = $ TNYWR. In $\triangle LMN$, point S lies between M and N and $\overline{LS} \perp \overline{MN}$. Given that $LN = 150$, $\sin(\angle SNL) = T$, and $\mathrm{m}\angle SML = 30°$, compute LM.

6. Let $T = $ TNYWR, and let $K = T - 50$. David drives a total of L miles at a constant speed of K miles per hour, and this trip lasted for 24 minutes. Compute the value of L.

7. Let $T = $ TNYWR. A sphere with radius r inches has volume $(T + 4) \cdot 12\pi$ cubic inches. Compute r.

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15. Let $a_1 = 50$, $a_2 = 50 + 51$, $a_3 = 50 + 51 + 52$, and in general, a_n is the sum of n consecutive integers, the least of which is 50. Compute the ones digit (i.e., the rightmost digit) of a_{24}.

14. Let $T = $ TNYWR. To celebrate ARML's 50^{th} anniversary, Everett went on a safari. He observed several elk and lions. Each elk has two antlers and four legs, and each lion has four legs. Everett counted the total numbers of antlers and legs. The sum of the total numbers of antlers and legs was 436. The sum of the total numbers of antlers and lion's legs was $50T$. Compute the total number of animals (i.e., elk and lions) that Everett observed.

13. Let $T = $ TNYWR. In equilateral triangle ARM, point L is the centroid (i.e., the point where the three medians intersect). Point S is the midpoint of $\overline{AL}$, and $[SML] = T$. Compute $\frac{[ARML]}{[SAM]}$ (that is, the ratio of the area of [non-convex] quadrilateral $ARML$ to the area of triangle SAM.)

12. Let $T = $ TNYWR. On May 30, 2025, suppose the price of 1 ounce of gold was $50(50 + T)$ dollars. Suppose further that on June 2 (the following business day), the price of 1 ounce of gold will increase by $25T$ dollars. Compute the number of dollars for the price of $\frac{1}{100}$ ounces of gold on June 2.

11. Let $T = $ TNYWR. The number $\log T - \log 4$ is equal to $\log x$. Compute the integer y such that $|y \log x - \log 50|$ is as small as possible.

10. Let $T = $ TNYWR. The sum $\binom{50}{T} + \binom{50}{T+1}$ equals $\binom{51}{K}$, where $K \neq T + 1$. Compute K.

9. Let $T = $ TNYWR, and let $K = T + 2$. Let $P(x) = 50x^{50} - Kx^{49} + 50x^{48} - Kx^{47} + \cdots - Kx + 50$, where the coefficients alternate between 50 and $-K$. Compute the sum of the product of the roots and the sum of the roots of $P(x)$.

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8. Let J be the number you will receive from position 7, and let K be the number you will receive from position 9. Let $j_1, j_2, j_3, \ldots$ be the arithmetic sequence defined by $j_1 = J$, $j_2 = J + 1$, $j_3 = J + 2$, etc. Let $k_1, k_2, k_3, \ldots$ be the arithmetic sequence defined by $k_1 = 50 - K$, $k_2 = 49 - K$, $k_3 = 48 - K$, etc. Compute $j_{50} + k_{50}$.

13 Super Relay Answers

1. 12

2. 25

3. 10

4. $\frac{1}{3}$

5. 100

6. 20

7. 6

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15. 6

14. 92

13. 4

12. 28

11. 2

10. 48

9. 2

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8. 54

14 Super Relay Solutions

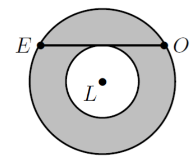
Problem 1. Compute the number of lattice points (points with integer coordinates) that lie on the graph of $x^2 + y^2 = 50$.

Solution 1. Note that if (a, b) lies on the graph of $\mathcal{G} : x^2 + y^2 = 50$, then so do the points $(-a, b)$, $(a, -b)$, and $(-a, -b)$. Because $1^2 + 7^2 = 7^2 + 1^2 = 5^2 + 5^2 = 50$, the points $P(1, 7)$, $Q(7, 1)$, and $R(5, 5)$ in the first quadrant all lie on $\mathcal{G}$. Because each point P, Q, R gives rise to three other "companion points" (obtained by reflection across the coordinate axes or origin), this is a total of $4 + 4 + 4 = \mathbf{12}$ points.

Problem 2. Let T = TNYWR. For all nonnegative real numbers N and some value of K, the equation $\sqrt[K]{N^T} = \sqrt[50]{N^{24}}$ is an identity. Compute the numerical value of K.

Solution 2. The given information implies that $\frac{T}{K} = \frac{24}{50} = \frac{12}{25}$. With $T = 12$, it follows that $K = \mathbf{25}$.

Problem 3. Let T = TNYWR. As shown, two circles are centered at point L. Points E and O lie on the larger circle so that chord $\overline{EO}$ is tangent to the smaller circle. The positive difference in the areas of the circles (shaded in the diagram) is $T\pi$. Compute EO.



Solution 3. Let r and R be the respective radii of the smaller and larger circles, and let S be the point in which $\overline{EO}$ is tangent to the smaller circle centered at L. Then $\overline{LS} \perp \overline{EO}$ and S is the midpoint of $\overline{EO}$. Hence $OS^2 + LS^2 = LO^2$, so $OS^2 = R^2 - r^2$. Thus $T = R^2 - r^2 = OS^2$ because the difference in the areas of the circles is $(R^2 - r^2)\pi$. With $T = 25$, $OS = 5$, and $EO = 2 \cdot OS = \mathbf{10}$.

Problem 4. Let T = TNYWR. To celebrate the 50^{th} ARML, Renee baked $5T$ chocolate cookies and Richard baked $10T$ snowball cookies. Then Renee ate one of her cookies and Richard ate two of his cookies. One of the remaining cookies was then randomly selected. Compute the probability that it was a chocolate cookie.

Solution 4. After the 3 cookies have been eaten, there are $5T - 1$ chocolate cookies and $10T - 2$ snowball cookies that remain. The desired probability is therefore $\frac{5T-1}{(5T-1)+(10T-2)} = \frac{1\cdot(5T-1)}{3\cdot(5T-1)} = \mathbf{\frac{1}{3}}$ (independent of T).

Problem 5. Let T = TNYWR. In $\triangle LMN$, point S lies between M and N and $\overline{LS} \perp \overline{MN}$. Given that $LN = 150$, $\sin(\angle SNL) = T$, and $m\angle SML = 30^\circ$, compute LM.

Solution 5. Note that $\sin(\angle SNL) = \frac{SL}{150} = T$, and from $30^\circ{-}60^\circ{-}90^\circ$ triangle SML, $ML = 2 \cdot SL = 300T$. With $T = \frac{1}{3}$, the answer is $\mathbf{100}$.

Problem 6. Let T = TNYWR, and let $K = T - 50$. David drives a total of L miles at a constant speed of K miles per hour, and this trip lasted for 24 minutes. Compute the value of L.

Solution 6. Converting, the number of hours in 24 minutes is $\frac{24}{60} = 0.2$. Thus $L = 0.2K$, and with $T = 100$, $K = 50$, and $L = \mathbf{20}$.

Problem 7. Let T = TNYWR. A sphere with radius r inches has volume $(T + 4) \cdot 12\pi$ cubic inches. Compute r.

Solution 7. Setting $\frac{4}{3}\pi r^3$ equal to $(T+4)\cdot 12\pi$ implies $r^3 = 9(T+4)$. With $T = 20$, $r^3 = 216$, so $r = \mathbf{6}$.

Problem 15. Let $a_1 = 50$, $a_2 = 50+51$, $a_3 = 50+51+52$, and in general, a_n is the sum of n consecutive integers, the least of which is 50. Compute the ones digit (i.e., the rightmost digit) of a_{24}.

Solution 15. Because each of the n terms in the sum for a_n is "offset" by 50 (which thus contributes 0 to the ones digit of a_n), it follows that the desired digit is the ones digit of the sum $1+2+\cdots+(n-1)$. The ones digit of a_{24} is therefore the ones digit of the sum $1+2+\cdots+23 = \frac{23\cdot 24}{2} = 276$, so the answer is $\mathbf{6}$.

Problem 14. Let $T = \text{TNYWR}$. To celebrate ARML's 50^{th} anniversary, Everett went on a safari. He observed several elk and lions. Each elk has two antlers and four legs, and each lion has four legs. Everett counted the total numbers of antlers and legs. The <u>sum</u> of the total numbers of antlers and legs was 436. The sum of the total numbers of antlers and <u>lion's legs</u> was $50T$. Compute the total number of animals (i.e., elk and lions) that Everett observed.

Solution 14. Let E and L be the respective numbers of elk and lions that Everett observed. The given conditions imply that $2E+(4E+4L) = 436$ and $2E+4L = 50T$. Solving this system yields $E = 109 - \frac{25}{2}T$. With $T = 6$, it follows that $E = 34$ and $L = 58$. Thus $E + L = \mathbf{92}$.

Problem 13. Let $T = \text{TNYWR}$. In equilateral triangle ARM, point L is the centroid (i.e., the point where the three medians intersect). Point S is the midpoint of $\overline{AL}$, and $[SML] = T$. Compute $\dfrac{[ARML]}{[SAM]}$ (that is, the ratio of the area of [non-convex] quadrilateral $ARML$ to the area of triangle SAM.)

Solution 13. Let $[ARM] = 6K$. Using the fact that $[ARL] = [MLR] = [AML]$, it follows that $[AML] = 2K$. Because S is the midpoint of $\overline{AL}$, it follows that $[SAM] = [SML]$. Thus $[SAM] = \frac{1}{2}[SML] = \frac{1}{2}[AML] = K$. Note also that $[ARML] = [AML] + [MLR] = 2K + 2K = 4K$. Thus $\dfrac{[ARML]}{[SAM]} = \dfrac{4K}{K} = \mathbf{4}$ (independent of T).

Problem 12. Let $T = \text{TNYWR}$. On May 30, 2025, suppose the price of 1 ounce of gold was $50(50+T)$ dollars. Suppose further that on June 2 (the following business day), the price of 1 ounce of gold will increase by $25T$ dollars. Compute the number of dollars for the price of $\dfrac{1}{100}$ ounces of gold on June 2.

Solution 12. In terms of T, the answer is $\frac{1}{100}\big(50(50+t)+25T\big) = 25 + \frac{3T}{4}$. With $T = 4$, this is $\mathbf{28}$.

Problem 11. Let $T = \text{TNYWR}$. The number $\log T - \log 4$ is equal to $\log x$. Compute the <u>integer</u> y such that $|y\log x - \log 50|$ is as small as possible.

Solution 11. Using logarithm identities, $\log T - \log 4 = \log\frac{T}{4} = x$, thus $|y\log x - \log 50| = \left|\log\left(\frac{T}{4}\right)^y - \log 50\right|$. With $T = 28$, the absolute value expression becomes $|\log 7^y - \log 50|$. Noting that $7^2 \approx 50$ and that $7^1 < 7^2 < 7^3$, the desired value of y is $\mathbf{2}$.

Problem 10. Let $T = \text{TNYWR}$. The sum $\dbinom{50}{T} + \dbinom{50}{T+1}$ equals $\dbinom{51}{K}$, where $K \neq T+1$. Compute K.

Solution 10. Recalling Pascal's Triangle, it can be verified algebraically that $\binom{50}{T} + \binom{50}{T+1} = \binom{51}{T+1}$. Because $K \neq T+1$, use the symmetry identity of binomial coefficients to yield that $\binom{51}{T+1} = \binom{51}{51-(T+1)} = \binom{51}{50-T}$. Thus $K = 50 - T$. With $T = 2$, the answer is **48**.

Problem 9. Let $T = \text{TNYWR}$, and let $K = T + 2$. Let $P(x) = 50x^{50} - Kx^{49} + 50x^{48} - Kx^{47} + \cdots - Kx + 50$, where the coefficients alternate between 50 and $-K$. Compute the <u>sum</u> of the product of the roots and the sum of the roots of $P(x)$.

Solution 9. Using Vieta's Formulas, the sum of the roots of $P(x)$ is $-\left(\frac{-K}{50}\right) = \frac{K}{50}$, and the product of the roots of $P(x)$ is $\frac{50}{50} \cdot (-1)^{50} = 1$. With $T = 48$, $K = 50$, and the desired sum is $\frac{K}{50} + 1 = \mathbf{2}$.

Problem 8. Let J be the number you will receive from position 7, and let K be the number you will receive from position 9. Let $j_1, j_2, j_3, \ldots$ be the arithmetic sequence defined by $j_1 = J$, $j_2 = J + 1$, $j_3 = J + 2$, etc. Let $k_1, k_2, k_3, \ldots$ be the arithmetic sequence defined by $k_1 = 50 - K$, $k_2 = 49 - K$, $k_3 = 48 - K$, etc. Compute $j_{50} + k_{50}$.

Solution 8. Note that in general, $j_n = (n-1) + J$ and $k_n = (51 - n) - K$. Thus $j_{50} + k_{50} = (49 + J) + (1 - K) = 50 + J - K$. With $J = 6$ and $K = 2$, the final answer is $50 + 6 - 2 = \mathbf{54}$.

15 Tiebreaker Round

Problem 1. Suppose m and n are integers that satisfy the system

$$m + n^2 + mn = 497$$
$$n + 2m^2 = 546.$$

Compute $m + n$.

Problem 2. Compute the lattice point (a, b) closest to the origin for which $(a - b)\sqrt{a + b} = 999$.

Problem 3. Let $S = \{1, 2, 3, 4, 5, 6\}$. Compute the number of nonempty subsets T of S for which $\dfrac{\min T}{|T|} \in S$.

16 Tiebreaker Round Answers

Answer 1. -15

Answer 2. $(96, -15)$

Answer 3. 15

17 Tiebreaker Round Solutions

Problem 1. Suppose m and n are integers that satisfy the system

$$\begin{aligned} m + n^2 + mn &= 497 \\ n + 2m^2 &= 546. \end{aligned}$$

Compute $m + n$.

Solution 1. The problem statement implies

$$(n - m) + (2m^2 - n^2 - mn) = 546 - 497 = 49.$$

Note that $(m - n)$ can be factored from the left-hand side:

$$(m - n)(2m + n - 1) = 49.$$

So $m - n$ and $2m + n - 1$ must be (possibly negative) factors of 49 that multiply to 49. Furthermore, the sum of the two factors is $3m - 1$ and thus must be equivalent to 2 mod 3. So $(m - n, 2m + n - 1)$ is one of $(1, 49)$, $(7, 7)$, or $(49, 1)$. In each case, (m, n) is determined by solving the systems of equations implied by the first two columns of the following table.

$m - n$	$2m + n - 1$	m	n
1	49	17	16
7	7	5	-2
49	1	17	-32

Now substitute back in to the original system. In the pair $(m, n) = (5, -2)$, note that m and $|n|$ are too small to satisfy the original system. If $m = 17$, then $2m^2 = 578$, so n must be -32 for the second of the given equations to hold. Then the first given equation also holds, so the answer is $17 + (-32) = \mathbf{-15}$.

Problem 2. Compute the lattice point (a, b) closest to the origin for which $(a - b)\sqrt{a + b} = 999$.

Solution 2. First note that because a and b are integers, $a + b$ must be a perfect square. Moreover, because the factor of $(a - b)$ is multiplied by a positive integer, it follows that $a - b$ is also a positive integer. Now consider the following table.

$a - b$	$a + b$
1	999^2
3	333^2
9	111^2
27	37^2
37	27^2
111	9^2
333	3^2
999	1^2

The problem is to find the ordered pair (a, b) that minimizes $a^2 + b^2$. Now consider the pair of equations $a - b = 333$ and $a + b = 9$. This is satisfied by $(a, b) = (171, -162)$. Note that unless $a - b = 111$ and $a + b = 81$, each of the other pairs of equations implied by the above table lead to $a \geq 171$ and $|b| \geq 162$. On the other hand, the pair of equations $a - b = 111$ and $a + b = 81$ is satisfied by $\mathbf{(96, -15)}$, which is the desired point because $96^2 + (-15)^2 < 171^2 + 162^2$.

Problem 3. Let $S = \{1, 2, 3, 4, 5, 6\}$. Compute the number of nonempty subsets T of S for which $\dfrac{\min T}{|T|} \in S$.

Solution 3. Suppose that $|T| = 1$. Then there is only one element in the subset T, and that element is the minimum of the subset. Thus there are 6 such subsets T.

Suppose that $|T| = 2$. Then the minimum of T can be either 2 or 4. If $\min T = 2$, then there are 4 other elements that could possibly be in T. If instead $\min T = 4$, then there are 2 other elements that could possibly be in T. Thus there are $4 + 2 = 6$ subsets T of this type.

Suppose that $|T| = 3$. Then the minimum of T can be only 3. The other two possible elements of T can be chosen in $\binom{3}{2} = 3$ ways, so there are 3 subsets T of this type.

If $|T| > 3$, then $\min T > 3$, but then this implies that $|T| \leq 3$, which contradicts the assumption that $|T| > 3$.

Thus the final answer is $6 + 6 + 3 = \mathbf{15}$.